

Mixed orthogonal arrays, k -dimensional M -part Sperner multi-families, and full multi-transversals

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This paper is dedicated to the memory of Professor Rudolf Ahlswede.

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Abstract

Aydinian *et al.* [J. Combinatorial Theory A **118**(2)(2011), 702–725] substituted the usual BLYM inequality for L -Sperner families with a set of M inequalities for $(m_1, m_2, \dots, m_M; L_1, L_2, \dots, L_M)$ type M -part Sperner families and showed that if all inequalities hold with equality, then the family is homogeneous. Aydinian *et al.* [Australasian J. Comb. **48**(2010), 133–141] observed that all inequalities hold with equality if and only if the transversal of the Sperner family

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corresponds to a simple mixed orthogonal array with constraint M , strength $M - 1$, using $m_i + 1$ symbols in the i^{th} column. In this paper we define k -dimensional M -part Sperner multi-families with parameters $L_P : P \in \binom{[M]}{k}$ and prove $\binom{M}{k}$ BLYM inequalities for them. We show that if $k < M$ and all inequalities hold with equality, then these multi-families must be homogeneous with profile matrices that are strength $M - k$ mixed orthogonal arrays. For $k = M$, homogeneity is not always true, but some necessary conditions are given for certain simple families. Following the methods of Aydinian *et al.* [Australasian J. Comb. **48**(2010), 133–141], we give new constructions to simple mixed orthogonal arrays with constraint M , strength $M - k$, using $m_i + 1$ symbols in the i^{th} column. We extend the convex hull method to k -dimensional M -part Sperner multi-families, and allow additional conditions providing new results even for simple 1-part Sperner families.

1 Notations

We will use $[n] = \{1, 2, \dots, n\}$ and $[n]^* = \{0, 1, \dots, n - 1\}$, and let $\binom{X}{\ell}$ denote the family of all ℓ -element subsets of the set X .

We will talk about multisets, where every element appears with some positive integer multiplicity. We will use the notation $\overline{\{\cdot\}}$ to emphasize that we talk about a multiset. If \mathcal{A} is a multiset, then the support set $\text{supp}(\mathcal{A})$ of \mathcal{A} is the simple set containing all elements of \mathcal{A} . We denote the multiplicity of x in a multiset \mathcal{A} by $\#[x, \mathcal{A}]$. Clearly, $x \notin \mathcal{A}$ iff $\#[x, \mathcal{A}] = 0$.

For shortness, for multisets \mathcal{A} and simple sets \mathcal{B} we will use $\mathcal{A} \in \mathcal{B}$ to denote $\text{supp}(\mathcal{A}) \subseteq \mathcal{B}$, i.e. the event that all the elements of \mathcal{A} are elements of \mathcal{B} . If $P(\cdot)$ is a proposition and k_C are non-negative integers, then $\overline{\{C^{k_C} : P(C)\}}$ denotes the multiset we obtain by taking all objects C with multiplicity k_C that satisfy $P(\cdot)$. Clearly, if \mathcal{A} is a multiset, then $\overline{\{C^{\#[C, \mathcal{A}]} : P(C)\}}$ will only contain elements of \mathcal{A} .

If \mathcal{A}_i are a multi-families of sets and $P(\cdot)$ is a Boolean polynomial on ℓ sets and k_{C_1, \dots, C_ℓ} are non-negative integers, then $\overline{\{C^{k_{C_1, \dots, C_\ell}} : C = P(C_1, \dots, C_\ell)\}}$ denotes the multiset where every C appears with multiplicity $\sum_{(C_1, \dots, C_\ell)} k_{C_1, \dots, C_\ell}$ where the sum is taken over all different ℓ -tuples (C_1, \dots, C_ℓ) for which $C = P(C_1, \dots, C_\ell)$.

For a multiset \mathcal{A} , the *size* or *cardinality* of \mathcal{A} is $|\mathcal{A}| = \sum_{x \in \mathcal{A}} \#[x, \mathcal{A}]$.

We use \uplus to denote disjoint unions of multisets; if \mathcal{A} and \mathcal{B} are multisets, then $\mathcal{A} \uplus \mathcal{B}$ denotes the multiset obtained by $\#[x, \mathcal{A} \uplus \mathcal{B}] = \#[x, \mathcal{A}] + \#[x, \mathcal{B}]$. Clearly, if \mathcal{A} and \mathcal{B} are disjoint (simple) sets, then \uplus is the usual (disjoint) union.

For multisets \mathcal{A} and \mathcal{B} , $\mathcal{A} \cup \mathcal{B}$ denotes the multiset obtained by $\#[x, \mathcal{A} \cup \mathcal{B}] = \max(\#[x, \mathcal{A}], \#[x, \mathcal{B}])$.

For multisets \mathcal{A} and \mathcal{B} , $\mathcal{A} \cap \mathcal{B}$ denotes the multiset obtained by $\#[x, \mathcal{A} \cap \mathcal{B}] = \min(\#[x, \mathcal{A}], \#[x, \mathcal{B}])$.

For multisets \mathcal{A} and \mathcal{B} , $\mathcal{A} \setminus \mathcal{B}$ denotes the multiset obtained by $\#[x, \mathcal{A} \setminus \mathcal{B}] = \max(0, \#[x, \mathcal{A}] - \#[x, \mathcal{B}])$.

A \mathcal{B} multiset of subsets of X is a *multichain* of length $|\mathcal{B}|$, if the elements of \mathcal{B} are pairwise comparable (i.e. the different elements of \mathcal{B} form a chain in the usual sense, and elements may occur with higher multiplicity then 1).

A multiset \mathcal{B} is called an *antichain* if it is a set forming an antichain. Antichains are always simple sets.

Finally, if \mathcal{F} is a multiset and $k(F)$ is a real-valued function on $\text{supp}(\mathcal{F})$, then we use the notation

$$\sum_{F \in \mathcal{F}} k(F) := \sum_{F \in \text{supp}(\mathcal{F})} k(F) \cdot \#[F, \mathcal{F}].$$

2 Definitions: k -dimensional multi-transversals and mixed orthogonal arrays

Let us be given $1 \leq n_1, \dots, n_M$, a $k \in [M]$, and set for the rest of the paper $\pi_M = \prod_{i=1}^M [n_i]^\star$. For each $P \in \binom{[M]}{k}$ let us be given an integer L_P such that $1 \leq L_P$. A $\mathcal{T} \subseteq \pi_M$ is called a *k -dimensional multi-transversal*¹ on π_M with these parameters if for every $P \in \binom{[M]}{k}$, fixing $b_j \in [n_j]^\star$ arbitrarily for every $j \in [M] \setminus P$, we have that

$$\left| \overline{\left\{ (i_1, \dots, i_M)^{\#[(i_1, \dots, i_M), \mathcal{T}]} : i_j = b_j \text{ for all } j \in [M] \setminus P \right\}} \right| \leq L_P. \quad (1)$$

If we want to emphasize that \mathcal{T} is a set and not a multiset (i.e. every element of \mathcal{T} has multiplicity 1), then we call it a *k -dimensional transversal* or a *k -dimensional simple transversal*.

It is easy to see that if \mathcal{T} is a k -dimensional multi-transversal, then we have the inequalities

$$\forall P \in \binom{[M]}{k} \quad |\mathcal{T}| \leq L_P \prod_{j \notin P} n_j. \quad (2)$$

¹This concept is different from the *transversal design* in [13] even for the simple transversals.

A k -dimensional multi-transversal is called *full*, if equality holds for at least one inequality set by a $P \in \binom{[M]}{k}$. It is clear from the definitions that equality in one inequality (i.e. having a full transversal) implies equalities in all inequalities iff

$$\frac{1}{L_P} \prod_{j \in P} n_j \text{ does not depend on the choice of } P. \quad (3)$$

The k -dimensional multi-transversals above have intimate connection to mixed orthogonal arrays. Consider sets S_i of n_i symbols ($i = 1, \dots, M$) and consider an $N \times M$ matrix T , whose the i^{th} column draws its elements from the set S_i . This matrix is called a *mixed* (or *asymmetrical*) *orthogonal array* or *MOA* (the notion of *orthogonal array with variable numbers of symbols* is also used), of *strength* d , *constraint* M and *index set* \mathbb{L} , if for any choice of d different columns j_1, \dots, j_d each sequence $(a_{j_1}, \dots, a_{j_d}) \in S_{j_1} \times \dots \times S_{j_d}$ appears exactly $\lambda(j_1, \dots, j_d) \in \mathbb{L}$ times after deleting the other $M - d$ columns. In the case of equal symbol set sizes (and therefore constant λ) we have the classical definition of orthogonal arrays. A (mixed) orthogonal array is *simple*, if the matrix T has no repeated rows. The following proposition easily follows from the definitions.

Proposition 2.1. *If the parameters $n_1, \dots, n_M, \{L_P : P \in \binom{[M]}{k}\}$ satisfy the condition (3), then any full k -dimensional multi-transversal is a MOA with symbol sets $S_i = [n_i]^*$, of constraint M , strength $M - k$, and index set $\mathbb{L} = \{L_P : P \in \binom{[M]}{k}\}$, with $\lambda(j_1, \dots, j_{M-k}) = L_{[M] \setminus \{j_1, \dots, j_{M-k}\}}$. Furthermore, if the transversal is simple, then so is the MOA.*

Moreover, if a MOA T is given with symbol sets S_i , (where $n_i = |S_i|$), of constraint M , strength d , with an index set \mathbb{L} , then T corresponds to a full $(M - d)$ -dimensional multi-transversal with parameters n_i and $L_P = \lambda([M] \setminus P)$. Furthermore, if T is simple, then so is the corresponding multi-transversal.

Orthogonal arrays were introduced by Rao [16, 17], the terminology was introduced by Bush [4, 5]. Cheng [6] seems to be the first author to consider MOAs. MOAs are widely used in planning experiments. The standard reference work for (mixed) orthogonal arrays is the monograph of Hedayat, Sloane and Stufken [13]. Constructions for MOAs usually use finite fields and few MOAs of strength > 2 are known.

An alternative formulation to k -dimensional (simple) transversals is the following: a set of length M codewords from π_M , such that for every $P \in \binom{[M]}{k}$ set of character positions, if the characters are prescribed in any way

for the $i \notin P$ character positions, at most L_P of our codewords show all the prescribed values. In particular, if L_P is identically 1, then a k -dimensional transversal is a code of minimum Hamming distance $k + 1$ (see [18]).

Also, k -dimensional transversals are *packing arrays* and their complements are *covering arrays* (for the definitions, see [13]).

3 k -dimensional M -part Sperner multi-families

Let us be given an underlying set X of cardinality n (often just $X = [n]$), and a fixed partition X_1, \dots, X_M of X with $|X_i| = m_i$. Set $n_i = m_i + 1$ (this convention will be used throughout the paper from now on).

Assume that \mathcal{C}_i is a (simple) chain in the subset lattice of X_i , for $i \in P$, where $P \subseteq [M]$. We define the product of these chains as

$$\prod_{i \in P} \mathcal{C}_i = \left\{ \biguplus_{i \in P} A_i : A_i \in \mathcal{C}_i \right\}.$$

Let us be given for every $P \in \binom{[M]}{k}$ a positive integer L_P .

We call a multi-family of subsets of X , \mathcal{F} , a *k -dimensional M -part Sperner multi-family with parameters $\{L_P : P \in \binom{[M]}{k}\}$* , if for all $P \in \binom{[M]}{k}$, for all (simple) chains \mathcal{C}_j in X_j ($j \in P$) and for all fixed sets $D_i \subseteq X_i$ ($i \notin P$) we have that

$$\left| \overline{\left\{ F^{\# [F, \mathcal{F}]} : \left(F \cap \biguplus_{j \in P} X_j \right) \in \prod_{j \in P} \mathcal{C}_j, \forall i \in [M] \setminus P \ X_i \cap F = D_i \right\}} \right| \leq L_P.$$

A *k -dimensional M -part Sperner family* or a *simple k -dimensional M -part Sperner family* \mathcal{F} is a Sperner multi-family where $\# [F, \mathcal{F}] \in \{0, 1\}$. For simple families, for $k = 1$ we get back the concept of M -part $(m_1, \dots, m_M; L_1, \dots, L_M)$ -Sperner families from [1], and restricting further with $M = 1$, we get back the concept of the classical L -Sperner families.

The *profile vector* of a subset F of X is the M -dimensional vector

$$(|F \cap X_1|, \dots, |F \cap X_j|, \dots, |F \cap X_M|) \in \pi_M.$$

The *profile matrix* $\mathbb{P}(\mathcal{F}) = (p_{i_1, \dots, i_M})_{(i_1, \dots, i_M) \in \pi_M}$ of a multi-family \mathcal{F} of subsets of X is an M -dimensional matrix, whose entries count with multiplicity the elements of \mathcal{F} with a given profile vector:

$$p_{i_1, i_2, \dots, i_M} = \left| \overline{\left\{ F^{\# [F, \mathcal{F}]} : \forall j \ |F \cap X_j| = i_j \right\}} \right|.$$

A multi-family \mathcal{F} of subsets of X is called *homogeneous*, if the profile vector of a set determines the multiplicity of the set in \mathcal{F} . In a homogeneous multi-family \mathcal{F} , we have that for each profile vector (i_1, \dots, i_M) there is a non-negative integer r_{i_1, \dots, i_M} such that $p_{i_1, i_2, \dots, i_M} = r_{i_1, \dots, i_M} \prod_{j=1}^M \binom{m_j}{i_j}$. For simple families, $r_{i_1, \dots, i_M} \in \{0, 1\}$, and this concept of homogeneity simplifies to the usual concept.

Given a homogeneous k -dimensional M -part Sperner multi-family \mathcal{F} with parameters $\{L_P : P \in \binom{[M]}{k}\}$, we observe that the multiset containing each (i_1, \dots, i_M) with multiplicity r_{i_1, \dots, i_M} is a k -dimensional multi-transversal with these parameters, and every k -dimensional multi-transversal comes from a homogeneous k -dimensional M -part Sperner multi-family. The multi-family is a (simple) family precisely when the corresponding multi-transversal is in fact a simple transversal.

4 New Sperner type results

In Sections 4, 5 and 7 we do not break the narrative with lengthy proofs and leave those to Sections 8, 9 and 10. We start with the following:

Theorem 4.1. [BLYM inequalities] *Given a k -dimensional M -part Sperner multi-family \mathcal{F} with parameters $\{L_P : P \in \binom{[M]}{k}\}$, the following inequalities hold:*

$$\forall P \in \binom{[M]}{k} \quad \sum_{(i_1, \dots, i_M) \in \pi_M} \frac{p_{i_1, \dots, i_M}}{\prod_{j=1}^M \binom{m_j}{i_j}} \leq \frac{L_P}{\prod_{j \in P} n_j} \prod_{j=1}^M n_j. \quad (4)$$

For simple families, the special case of this theorem for $k = 1$ was found by Aydinian, Czaparka, P. L. Erdős, and Székely in [1], Theorem 6.1. The special case for $M = 1$ was first in print in [10], and the special case $L = M = 1$ is the Bollobás–Lubell–Meshalkin–Yamamoto (BLYM) inequality [3, 14, 15, 20]. Note that the single classical BLYM inequality has been substituted by a *family* of inequalities. Cases of equality can be characterized as follows:

Theorem 4.2. *Given integers $1 = k \leq M$ or $2 \leq k \leq M - 1$, let \mathcal{F} be a k -dimensional M -part Sperner multi-family with parameters $\{L_P : P \in \binom{[M]}{k}\}$ satisfying all inequalities in (4) with equality. Then the following are true:*

- (i) \mathcal{F} is homogeneous;

- (ii) $\frac{L_P}{\prod_{j \in P} n_j}$ does not depend on the choice of P ;
- (iii) the k -dimensional multi-transversal corresponding to \mathcal{F} is a MOA with symbol sets $S_i = [n_i]^\star$, of constraint M , strength $M - k$, and index set $\mathbb{L} = \{L_P : P \in \binom{[M]}{k}\}$, with $\lambda(j_1, \dots, j_{M-k}) = L_{[M] - \{j_1, \dots, j_{M-k}\}}$.

Any MOA, as described in (iii) is a k -dimensional multi-transversal on π_M with parameters $\{L_P : P \in \binom{[M]}{k}\}$, and it corresponds to the profile matrix of a homogeneous k -dimensional M -part Sperner multi-family \mathcal{F} with parameters $\{L_P : P \in \binom{[M]}{k}\}$ on a partitioned $(m_1 + \dots + m_M)$ -element underlying set, which satisfies all inequalities in (4) with equality.

Under this correspondence, simple k -dimensional M -part Sperner families correspond to simple MOAs.

Note that the last sentence is obvious and part (ii) follows directly from the conditions of the theorem.

The special case of this theorem for $k = 1$ and for simple families and simple transversals was found in [1], Theorem 6.2 but failed to mention (iii). Note also that (iii) turns trivial for $M = k = 1$, as the matrix in question has a single column. Conclusion (i) for the special case $L = k = M = 1$ restricted to simple families is known as the strict Sperner theorem, already known to Sperner [19]; for $M = 1$, $L > 1$, it was discovered by Paul Erdős [8]. However, Theorem 4.2 does not hold for $k = M \geq 2$, as the following example shows.

Example 4.3. Let $k = M \geq 2$ and $L_{[M]} = 1$ with $|X_i| = m_i$ for $i \in [M]$, and assume $m_M \geq 2$. For integers r, s with $1 \leq r \leq m_M - 1$ and $2 \leq s \leq \min(n_1, \dots, n_{M-1}, \binom{m_M}{r})$, consider a partition $\binom{X_M}{r} = \mathcal{B}_1 \uplus \dots \uplus \mathcal{B}_s$; and for each $j \in [M - 1]$, fix an s -element set $\{i_1^{(j)}, \dots, i_s^{(j)}\} \subseteq [n_j]^\star$. Define a k -dimensional k -part Sperner family \mathcal{F} as follows:

$$\mathcal{F} = \biguplus_{\ell=1}^s \left(\left(\prod_{j=1}^{M-1} \binom{X_j}{i_\ell^{(j)}} \right) \times \mathcal{B}_\ell \right).$$

This \mathcal{F} is not homogeneous, but by

$$\sum_{(i_1, \dots, i_M) \in \pi_M} \frac{p_{i_1, \dots, i_M}}{\prod_{j=1}^M \binom{m_j}{i_j}} = \sum_{\ell=1}^s \frac{|\mathcal{B}_\ell| \prod_{j=1}^{M-1} \binom{m_j}{i_\ell^{(j)}}}{\binom{m_M}{r} \prod_{j=1}^{M-1} \binom{m_j}{i_\ell^{(j)}}} = \frac{\sum_{\ell=1}^s |\mathcal{B}_\ell|}{\binom{m_M}{r}} = 1 = L_{[M]},$$

\mathcal{F} still satisfies (4) with (a single) equality.

The above example can be easily extended to $L_{[M]} > 1$. Although we did not characterize cases of equality in (4) for $k = M$, in the case $L_{[M]} = 1$ we are able to give a necessary condition for an M -dimensional M -part Sperner family to satisfy equality in (4).

Theorem 4.4. *Let \mathcal{F}' be a k -dimensional M -part Sperner family with $k = M$ and $L_{[M]} = 1$, satisfying the equality*

$$\sum_{E \in \mathcal{F}'} \frac{1}{\prod_{i=1}^k \binom{m_i}{|E \cap X_i|}} = 1. \quad (5)$$

Then for each $i \in [M]$, the trace $\mathcal{F}'_{X_i} := \{F \cap X_i : F \in \mathcal{F}'\}$ of \mathcal{F}' on X_i is a union of full levels of 2^{X_i} .

For the proof of Theorem 4.2 we need to prove a special case that is also a straightforward generalization of the BLYM for 1-part L -Sperner families, as stated below.

Lemma 4.5. *Let \mathcal{F} be a multi-family of subsets of $[n]$ containing no multichain of length $L + 1$. Then we have*

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq L, \quad (6)$$

with equality if and only if \mathcal{F} is homogeneous.

Proof. The inequality part follows from Theorem 4.1, $k = M = 1$. Suppose now we have equality in (6). We claim then that \mathcal{F} can be partitioned into L or less antichains. (In fact, this is the multiset analogue of the well-known dual version of Dilworth's Theorem.) We now mimick the inductive proof that works for simple families. For $L = 1$, \mathcal{F} has to be a simple family and the claim is exactly the strict Sperner Theorem. Let $L > 1$ and assume that the statement is true for all $1 \leq L' < L$. Consider the set \mathcal{F}_1 of maximal elements in \mathcal{F} (note that the multiplicity of each element in \mathcal{F}_1 is one). Then $\mathcal{F}_2 := \mathcal{F} \setminus \mathcal{F}_1$ contains no multichain of length L . Thus we have

$$\sum_{F \in \mathcal{F}_1} \frac{1}{\binom{n}{|F|}} \leq 1 \text{ and } \sum_{F \in \mathcal{F}_2} \frac{1}{\binom{n}{|F|}} \leq L - 1. \quad (7)$$

But we also have

$$L = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} = \sum_{F \in \mathcal{F}_1} \frac{1}{\binom{n}{|F|}} + \sum_{F \in \mathcal{F}_2} \frac{1}{\binom{n}{|F|}},$$

therefore equality holds in both inequalities at (7), and by the induction hypothesis both \mathcal{F}_1 and \mathcal{F}_2 are homogeneous. The lemma follows. \square

5 Convex hull of profile matrices of M -part multi-families

The vertices of the convex hull of profile matrices of different kind of families were described by P. L. Erdős, Frankl, and Katona [9], facilitating the optimization of linear functions of the entries of profile matrices of members of the family in question. P. L. Erdős and Katona [11] adapted the method for M -part Sperner families, and recently Aydinian, Czabarka, P. L. Erdős, and Székely adapted it for 1-dimensional M -part $(m_1, \dots, m_M; L_1, \dots, L_M)$ Sperner families. The purpose of this section to generalize these results for k -dimensional M -part Sperner multi-families, and even further.

Let $X = X_1 \uplus X_2 \uplus \dots \uplus X_M$ be a partition of the n -element underlying set X , where $|X_i| = m_i \geq 1$ and $m_1 + \dots + m_M = n$. Let \mathcal{F} be a multi-family of subsets of X . The profile matrix $\mathbb{P}(\mathcal{F}) := (p_{i_1, \dots, i_M})_{(i_1, \dots, i_M) \in \pi_M}$ can be identified with a point or its location vector in the Euclidean space \mathbb{R}^N , where $N = \prod_{j=1}^M n_j$.

Let $\alpha \subseteq \mathbb{R}^N$ be a finite point set. Let $\langle \alpha \rangle$ denote the *convex hull* of the point set, and $\varepsilon(\alpha) = \varepsilon(\langle \alpha \rangle)$ its *extreme points*. It is well-known that $\langle \alpha \rangle$ is equal to the set of all convex linear combinations of its extreme points.

Let \mathbb{A} be a family of multi-families of subsets of X . Let $\mu(\mathbb{A})$ denote the set of all profile-matrices of the multi-families in \mathbb{A} , i.e.

$$\mu(\mathbb{A}) = \{\mathbb{P}(\mathcal{F}) : \mathcal{F} \in \mathbb{A}\}.$$

Then the extreme points $\varepsilon(\mu(\mathbb{A}))$ are integer vectors and they are profile matrices of multi-families from \mathbb{A} .

In [11], P. L. Erdős and G.O.H. Katona developed a general method to determine the extreme points $\varepsilon(\mu(\mathbb{A}))$ for families of simple families. We adapt their results to a more general setting. Let $I \in \pi_M$. Let $T(I)$ denote the M -dimensional matrix, in which the entry $t_{i_1, \dots, i_M}(I) = \#[(i_1, \dots, i_M), I]$. Furthermore, let $S(I)$ be the M -dimensional matrix, in which $S_{i_1, \dots, i_M}(I) = t_{i_1, \dots, i_M}(I) \binom{m_1}{i_1} \dots \binom{m_M}{i_M}$. Recall that a multi-family of subsets of an M -partitioned underlying set is called *homogeneous*, if for any set, the sizes of its intersections with the partition classes already determine the (possibly 0) multiplicity with which the set belongs to the multi-family. It is easy to see that a homogeneous multi-family \mathcal{F} on X has $\mathbb{P}(\mathcal{F}) = S(I)$ for a certain multiset $I \in \pi_M$.

We say that \mathfrak{L} is a *product-permutation* of X , if the ordered n -tuple $\mathfrak{L} = (x_1, \dots, x_n)$ is a permutation of $X = X_1 \uplus X_2 \uplus \dots \uplus X_M$ such that $X_j = \{x_i : i = m_1 + \dots + m_{j-1} + 1, \dots, m_1 + \dots + m_j\}$ i.e. \mathfrak{L} is a juxtaposition of permutations of X_1, X_2, \dots, X_M , in this order. Furthermore, we

say that a subset $H \subseteq X$ is *initial with respect to \mathfrak{L}* , if for all $j = 1, 2, \dots, M$ we have

$$H \cap X_j = \left\{ x_{m_1 + \dots + m_{j-1} + 1}, \dots, x_{m_1 + \dots + m_{j-1} + |H \cap X_j|} \right\},$$

i.e. $H \cap X_j$ is an initial segment in the permutation of X_j . For an \mathcal{H} multi-family on X , define $\mathcal{H}(\mathfrak{L}) = \{H^{\# [H, \mathcal{H}]} : H \text{ is initial with respect to } \mathfrak{L}\}$. Similarly, for an \mathbb{A} family of multi-families on X , let $\mathbb{A}(\mathfrak{L}) := \{\mathcal{H}(\mathfrak{L}) : \mathcal{H} \in \mathbb{A}\}$.

Lemma 5.1 (cf. [11] Lemma 3.1). *Suppose that for a finite family \mathbb{A} of M -part multi-families the set $\mu(\mathbb{A}(\mathfrak{L}))$ does not depend on the choice of \mathfrak{L} . Then*

$$\mu(\mathbb{A}) \subseteq \left\langle \left\{ S(I) : I \in \pi_M \text{ with } T(I) \in \mu(\mathbb{A}(\mathfrak{L})) \right\} \right\rangle \quad (8)$$

holds.

The next theorem follows easily from this lemma:

Theorem 5.2 (cf. [11] Theorem 3.2). *Suppose that a finite family \mathbb{A} of M -part multi-families satisfies the following two conditions:*

$$\text{the set } \mu(\mathbb{A}(\mathfrak{L})) \text{ does not depend on } \mathfrak{L}, \text{ and} \quad (9)$$

$$\text{for all } I \in \pi_M, T(I) \in \mu(\mathbb{A}(\mathfrak{L})) \text{ implies } S(I) \in \mu(\mathbb{A}). \quad (10)$$

Then

$$\varepsilon(\mu(\mathbb{A})) = \varepsilon \left(\left\{ S(I) : I \in \pi_M, T(I) \in \mu(\mathbb{A}(\mathfrak{L})) \right\} \right). \quad (11)$$

Consequently, among the maximum size elements of \mathbb{A} , there are homogeneous ones, and the profile matrices of maximum size elements of \mathbb{A} are convex linear combinations of the profile matrices of homogeneous maximum size elements.

Proof. The identity

$$\langle \mu(\mathbb{A}) \rangle = \left\langle \left\{ S(I) : I \in \pi_M \text{ with } T(I) \in \mu(\mathbb{A}(\mathfrak{L})) \right\} \right\rangle$$

follows from (8) and (10). If two convex sets are equal, then so are their extreme points. \square

For any finite set Γ , a Γ -multiplicity constraint \mathbb{M}_Γ is

$$\mathbb{M}_\Gamma = \{(A_\gamma \geq 0, \{\alpha_{(i_1, \dots, i_M)}^\gamma \geq 0 : (i_1, \dots, i_M) \in \pi_M\}) : \gamma \in \Gamma\}.$$

We say that a multiset $\mathcal{F} \in X$ satisfies the Γ -multiplicity constraint \mathbb{M}_Γ , if

$$\forall \gamma \in \Gamma \quad \sum_{(i_1, \dots, i_M) \in \pi_M} \alpha_{i_1, \dots, i_M}^\gamma \cdot \max\{\#[F, \mathcal{F}] : \forall j \in [M] \ |F \cap X_j| = i_j\} \leq A_\gamma.$$

Analogously, a multiset $I \in \pi_M$ satisfies the Γ -multiplicity constraint \mathbb{M}_Γ , if

$$\forall \gamma \in \Gamma \quad \sum_{(i_1, \dots, i_M) \in \pi_M} \alpha_{i_1, \dots, i_M}^\gamma \cdot \#[(i_1, \dots, i_M), I] \leq A_\gamma.$$

It is easy to see that simple families can be characterized by the following condition: For all $(i_1, \dots, i_M) \in \pi_M$, $\max\{\#[F, \mathcal{F}] : \forall j \ |F \cap X_j| = i_j\} \leq 1$. This in turn can be written in the form of a Γ -multiplicity constraint by $\Gamma = \pi_M$, $A_\gamma = 1$, $\alpha_\lambda^\gamma = \delta_{\gamma, \lambda}$ using the Kronecker δ notation.

Theorem 5.3. *To the family \mathbb{A} of k -dimensional M -part Sperner multi-families with parameters L_P for $P \in \binom{[M]}{k}$ satisfying a fixed Γ -multiplicity constraint \mathbb{M}_Γ , Theorem 5.2 applies. In other words, all extreme points of $\mu(\mathbb{A})$ come from homogeneous multi-families.*

This theorem implies the results of [11] and [1] on the convex hull with one exception: there not just all extreme points came from homogeneous families, but all homogeneous families provided extreme points. This is not the case, however, for multi-families, but characterizing which homogeneous families are extreme is hopeless. For *simple families*, however, we can characterize these extreme points.

We say that an I k -dimensional M -part multi-transversal with Γ -multiplicity constraint \mathbb{M}_Γ is *lexicographically maximal (LEM)*, if the support set $\text{supp}(I) \subseteq \pi_M$ of the multiset I has an ordering $\vec{j}_1, \vec{j}_2, \dots, \vec{j}_s$, such that for every I^* k -dimensional M -part multi-transversal with Γ -multiplicity constraint \mathbb{M}_Γ , the following holds:

- (i) If $\vec{j}_1 \in \text{supp}(I^*)$, then $\#[\vec{j}_1, I] \geq \#[\vec{j}_1, I^*]$, and
- (ii) for every $1 \leq \ell \leq s-1$, if $\{\vec{j}_1, \vec{j}_2, \dots, \vec{j}_\ell\} \subseteq \text{supp}(I^*)$ and $\#[\vec{j}_h, I] = \#[\vec{j}_h, I^*]$ for $h = 1, 2, \dots, \ell$, then $\#[\vec{j}_{\ell+1}, I] \geq \#[\vec{j}_{\ell+1}, I^*]$.

Lemma 5.4. *For a family \mathbb{A} of k -dimensional M -part Sperner multi-families with parameters L_P for $P \in \binom{[M]}{k}$ satisfying a Γ -multiplicity constraint \mathbb{M}_Γ , all the profile matrices $S(I)$, where $I \in \pi_M$ is a LEM k -dimensional multi-transversal with the same Γ -multiplicity constraint, are extreme points of $\mu(\mathbb{A})$.*

For simple k -dimensional M -part Sperner families \mathcal{F} , i.e. when the Γ -multiplicity constraint \mathbb{M}_Γ includes the conditions $\max\{\#[F, \mathcal{F}] : \forall j \ |F \cap$

$|X_j| = i_j\} \leq 1$ for all $(i_1, \dots, i_M) \in \pi_M$, every I k -dimensional M -part Sperner multi-transversal with parameters L_P for $P \in \binom{[M]}{k}$ satisfying a Γ -multiplicity constraint \mathbb{M}_Γ has the LEM property. This finally derives the convex hull results of [1] and [11] from our results. Note, however, that the Γ -multiplicity constraint provides new results even for the classical $M = 1$ case. For completeness, we state explicitly our result for simple families.

Theorem 5.5. *The extreme points of the convex hull of profile matrices of all k -dimensional M -part simple Sperner families with a Γ -multiplicity constraint \mathbb{M}_Γ are exactly the profile matrices of the homogeneous families corresponding to k -dimensional M -part simple transversals with the same Γ -multiplicity constraint \mathbb{M}_Γ . Therefore, among the maximum size k -dimensional M -part Sperner families with a Γ -multiplicity constraint, there are homogeneous ones.*

6 Applications of the convex hull method

Although the previous section reduces the problem of finding the maximum size of such families to a "number" problem from a "set" problem, however, we assert that the problem is still "combinatorial" due to the complexity of transversals:

Problem 6.1. *For a $(t_1, \dots, t_M) \in \pi_M$, set the weight $W(t_1, \dots, t_M) = \prod_{i=1}^M \binom{m_i}{t_i}$. Find a set of codewords $C \subseteq \pi_M$ with the largest possible sum of weights, such that for every $P \in \binom{[M]}{k}$ set of character positions, if the characters are prescribed in any way for the $i \notin P$ character positions, at most L_P from C show all the prescribed values.*

In view of Theorem 5.5, Problem 6.1 is equivalent to finding maximum size k -dimensional M -part simple Sperner families. Recall that this problem is not solved even for the case $L = 1, k = 1, M \geq 3$ (see [1] for a survey of results). Note also that there are examples in [1] without a full 1-dimensional transversal defining a maximum size homogeneous family, unlike in the case $M = 2, L = 1$.

Our results allow us to prove that certain maximum size families must always be homogeneous.

Theorem 6.2. *Let $1 \leq k < M$ or $k = M = 1$. If every maximum size homogeneous k -dimensional M -part Sperner family (alternatively: Sperner multi-family) satisfies (4) with equality, then every maximum size k -dimensional M -part Sperner family (Sperner multi-family) is homogeneous.*

Proof. Fix a $P \in \binom{[M]}{k}$ and let $C = L_P \prod_{j \notin P} n_j$. By the assumptions, the value of C is independent of P . Let \mathcal{F} be a maximum size family/multi-family with profile matrix $\mathbb{P}(\mathcal{F}) = (p_{(i_1, \dots, i_M)})$. Let $\mathcal{G}_1, \dots, \mathcal{G}_s$ be an enumeration of all maximum size homogeneous families/multi-families, and let $I_1, \dots, I_s \subseteq \pi_M$ be the $(M-1)$ -dimensional transversals/multi-transversals for which $\mathbb{P}(\mathcal{G}_j) := (p_{(i_1, \dots, i_M)}^{(j)}) = S(I_j)$. By the assumptions for each $j \in [s]$ we have

$$\sum_{(i_1, \dots, i_M) \in \pi_M} \frac{p_{(i_1, \dots, i_M)}^{(j)}}{\prod_{\ell=1}^M \binom{m_\ell}{i_\ell}} = C.$$

By Theorems 5.3 and 5.5 we have $\lambda_j \geq 0$ such that $\sum_j \lambda_j = 1$ and $\mathbb{P}(\mathcal{F}) = \sum_{j=1}^s \lambda_j \mathbb{P}(\mathcal{G}_j)$. Therefore

$$\begin{aligned} \sum_{(i_1, \dots, i_M) \in \pi_M} \frac{p_{(i_1, \dots, i_M)}}{\prod_{\ell=1}^M \binom{m_\ell}{i_\ell}} &= \sum_{(i_1, \dots, i_M) \in \pi_M} \frac{\sum_{j=1}^s \lambda_j p_{(i_1, \dots, i_M)}^{(j)}}{\prod_{\ell=1}^M \binom{m_\ell}{i_\ell}} \\ &= \sum_{j=1}^s \left(\lambda_j \sum_{(i_1, \dots, i_M) \in \pi_M} \frac{p_{(i_1, \dots, i_M)}^{(j)}}{\prod_{\ell=1}^M \binom{m_\ell}{i_\ell}} \right) \\ &= C \sum_{j=1}^s \lambda_j = C, \end{aligned}$$

and \mathcal{F} is homogeneous by Theorem 4.2. \square

We state some simple results for the case when all parameters $L_P = 1$.

Theorem 6.3. *Consider the (simple) M -part families such that for all $E, F \in \mathcal{F}$, if $E \neq F$ then there is a $j \in [M]$ such that $E \cap X_j \neq F \cap X_j$. If \mathcal{F} is maximum size among these families, then*

$$|\mathcal{F}| = \prod_{i=1}^M \binom{m_i}{\lfloor m_i/2 \rfloor}. \quad (12)$$

\mathcal{F} is a maximum size homogeneous family precisely when $\mathbb{P}(\mathcal{F}) = S((\ell_1, \dots, \ell_M))$ where for each $i \in [M]$, $\ell_i \in \{\lfloor m_i/2 \rfloor, \lceil m_i/2 \rceil\}$. In particular, when all m_i are even, the maximum size family is unique and homogeneous.

Proof. A family \mathcal{G} satisfies the conditions precisely when it is a M -dimensional M -part Sperner family with $L = 1$. By Theorem 5.5, there are homogeneous families among such maximum size families. So let \mathcal{F} be a homogeneous

maximum size family. Then $\mathbb{P}(\mathcal{F}) = S(I)$ for some $I \subseteq \pi_M$. It follows from the conditions that $|I| = 1$, so $I = \{(i_1, \dots, i_M)\}$ and $|\mathcal{F}| = \prod_{j=1}^M \binom{m_j}{i_j}$. (12) follows, moreover the homogeneous maximum size families are precisely the ones listed in the theorem.

Since by Theorem 5.5 the profile matrix $\mathbb{P}(\mathcal{F})$ is the convex combination of the profile matrices of maximum size families, it follows that for m_i even the maximum size family is unique. \square

Note that it is easy to create a nonhomogeneous maximum size family when at least one of the m_i is odd along the lines of Example 4.3.

For the next result we will use the following, which follows easily by induction on K .

Lemma 6.4. *Let K, M be positive integers and for each $i \in [K]$ and $j \in [M]$ let a_{ij} be nonnegative reals such that $a_{1j} \geq a_{2j} \geq \dots \geq a_{K,j}$ and S_K denotes the set of permutations on $[K]$. Then*

$$\max \left\{ \sum_{\ell=1}^K \prod_{j=1}^M a_{\pi_j(\ell),j} : \forall j \in [M] \pi_j \in S_K \right\} = \sum_{\ell=1}^K \prod_{j=1}^M a_{\ell j}.$$

\square

Theorem 6.5. *Assume that $m_M = \min m_i$ and consider the $(M-1)$ -dimensional M -part Sperner families with parameters $L_{[M] \setminus \{i\}} = 1 : i \in [M]$. If \mathcal{F} is of maximum size amongst these families, then*

$$|\mathcal{F}| = \sum_{i=0}^{m_M} \prod_{j=1}^M \binom{m_j}{\lceil \frac{m_j}{2} \rceil + (-1)^i \lceil \frac{i}{2} \rceil}.$$

Moreover, if \mathcal{F} is a maximum size homogeneous family, then $\mathbb{P}(\mathcal{F}) = S(I)$ for some $I = \{(b_{i1}, \dots, b_{iM}) : i \in [n_M]^*\}$ where for each fixed $j \in [M]$ the b_{ij} are n_M different integers from $[n_j]^*$ such that $\binom{m_j}{b_{ij}} = \binom{m_j}{\lceil \frac{m_j}{2} \rceil + (-1)^i \lceil \frac{i}{2} \rceil}$.

If in addition $m_1 = \dots = m_M$, then all maximum size families are homogeneous.

Proof. Theorem 5.5 implies that amongst the maximum size families there are homogeneous ones. Let \mathcal{F} be a (not necessarily maximum size) homogeneous $(M-1)$ -dimensional M -part Sperner family with all parameters 1, and let I be the transversal for which $\mathbb{P}(\mathcal{F}) = S(I)$. Then if $\vec{i} = (i_1, \dots, i_M)$ and $\vec{i}' = (i'_1, \dots, i'_M)$ are elements of I such that for some $\ell \in [M]$ $i_\ell = i'_\ell$, we must have that $\vec{i} = \vec{i}'$. Therefore there is a $K \leq n_M$ such

that $I = \{(b_{i1}, \dots, b_{iM}) : i \in [K]^*\}$ where for each fixed $j \in [M]$ the b_{ij} are n_M different integers from $[n_j]^*$ and $|\mathcal{F}| = \sum_{\ell=1}^K \prod_{j=1}^M \binom{m_j}{b_{\ell j}}$. The statement about maximum size homogeneous families follows from Lemma 6.4 and the fact that

$$\binom{m_j}{\lceil \frac{m_j}{2} \rceil + (-1)^0 \lceil \frac{0}{2} \rceil} \geq \binom{m_j}{\lceil \frac{m_j}{2} \rceil + (-1)^1 \lceil \frac{1}{2} \rceil} \geq \dots \geq \binom{m_j}{\lceil \frac{m_j}{2} \rceil + (-1)^{m_j} \lceil \frac{m_j}{2} \rceil}.$$

The rest follows from Theorem 6.2. \square

7 New k -dimensional transversals and mixed orthogonal arrays

Aydinian, Czaparka, Engel, P. L. Erdős, and Székely [2] ran into MOAs as they faced the problem of constructing 1-dimensional full transversals for $M > 2$. Using the indicator function of the k -dimensional transversal in (1) instead of the transversal itself, it is easy to see that the existence of "fractional full k -dimensional transversal" is trivial. Therefore the construction problem of full k -dimensional transversals is a problem of integer programming. For $M = 2$, such construction was found [12] using matching theory, which does not apply for $M > 2$. [2] observed Proposition 2.1 for $k = 1$ (the property "simple" was assumed tacitly) and constructed 1-dimensional full transversals for any parameter set, and infinitely many MOAs with constraint M and strength $M - 1$. The key element of the construction was the elementary Lemma 7.1, which only uses properties of the fractional part $\langle x \rangle = x - \lfloor x \rfloor$ function of a real number x . This lemma will be heavily used again in this paper.

Lemma 7.1. [Engel's Lemma.] *Let n be a positive integer, μ, α, β be real numbers such that $0 < \mu$ and $0 \leq \beta \leq 1 - \mu$. Then*

$$\left| \left\{ i \in [n]^* : \left\langle \alpha + \frac{i}{n} \right\rangle \in [\beta, \beta + \mu) \right\} \right| \in \{\lfloor \mu n \rfloor, \lceil \mu n \rceil\}.$$

All our constructions for full k -dimensional transversals and simple MOAs are based on the following construction.

Construction 7.2. *For n_1, \dots, n_M positive integers, $0 < \mu \leq 1$ real, and*

$0 \leq \beta \leq 1 - \mu$, define

$$\mathbb{C}(n_1, \dots, n_M; \beta, \mu) := \left\{ (i_1, \dots, i_M) \in \pi_M : \left\langle \sum_{j=1}^M \frac{i_j}{n_j} \right\rangle \in [\beta, \beta + \mu) \right\}. \quad (13)$$

For the case $k = 1$, [2] showed that for any $i \in [M]$, any $L_i \in [n_i]$, any $0 < \mu \leq \min\{\frac{L_i}{n_i} : i \in [M]\}$, any $0 \leq \beta \leq 1 - \mu$, the construction in (13) is a 1-dimensional transversal for the given parameters, moreover, if $\mu = \min\{\frac{L_i}{n_i} : i \in [M]\}$, then this 1-dimensional transversal is full.

The following facts are almost immediate from the construction:

Proposition 7.3. *Let n_1, \dots, n_M be positive integers, $k \in [M]$, and $\{L_P : P \in \binom{[M]}{k}\}$ be given such that $1 \leq L_P \leq \prod_{i \in P} n_i$ are integers. If there is a $0 < \mu_0 \leq 1$ such that for each $0 \leq \beta \leq 1 - \mu_0$ the construction $\mathbb{C}(n_1, \dots, n_M; \beta, \mu_0)$ is a full k -dimensional transversal with these L_P parameters, then*

- (i) $\mathbb{C}(n_1, \dots, n_M; \beta, \mu)$ is a k -dimensional transversal with these parameters for every $0 < \mu < \mu_0$ and $0 \leq \beta \leq 1 - \mu$.
- (ii) π_M can be partitioned into $\lceil \frac{1}{\mu_0} \rceil$ k -dimensional transversals with these parameters, and $\lfloor \frac{1}{\mu_0} \rfloor$ of these are full.
- (iii) With $\alpha = \min\left\{\frac{L_P}{\prod_{i \in P} n_i} : P \in \binom{[M]}{k}\right\}$, we have $\lfloor \frac{1}{\alpha} \rfloor \leq \frac{1}{\mu_0} \leq \lceil \frac{1}{\alpha} \rceil$. In particular, if $\frac{1}{\alpha}$ is an integer, all k -dimensional transversals in the partition in (ii) are full.

Proof. (i) follows from the fact that for every β, μ in (i), exists a $0 \leq \beta' \leq 1 - \mu_0$, such that $[\beta, \beta + \mu) \subseteq [\beta', \beta' + \mu_0)$. (Here we did not use the fullness in the hypothesis.) For (ii), we use the fact that $[0, 1)$ can be partitioned into $\lceil \frac{1}{\mu_0} \rceil$ half-open intervals, $\lfloor \frac{1}{\mu_0} \rfloor$ of which has length μ_0 . Finally, (iii) follows from (2) and (ii). \square

We arrived at the following generalization of Engel's lemma (Lemma 7.1):

Lemma 7.4. *Let n_1, \dots, n_k be positive integers, $N = \text{lcm}(n_1, \dots, n_k)$, $K = \prod_{i=1}^k n_k$ and $\ell = \frac{K}{N}$. If α, β, μ are real numbers with $0 < \mu < 1$ and $0 \leq \beta \leq 1 - \mu$, then*

$$\left| \left\{ (i_1, \dots, i_k) \in \pi_M : \left\langle \alpha + \sum_{j=1}^k \frac{i_j}{n_j} \right\rangle \in [\beta, \beta + \mu) \right\} \right| \in \{\ell \lfloor \mu N \rfloor, \ell \lceil \mu N \rceil\}.$$

The proof of Lemma 7.4 is postponed to Section 10. Based on Lemma 7.4, the following theorem gives a sufficient criterion to use (13) to construct full k -dimensional transversals. For $k = 1$ it gives back the construction in [2].

We set a generic notation here for the rest of this section and Section 10. Let us be given $n_1, \dots, n_M \geq 1$ integers, a $k \in [M]$, and for every $P \in \binom{[M]}{k}$ let the integer L_P be given such that $1 \leq L_P \leq \prod_{i \in P} n_i$. For every $P \in \binom{[M]}{k}$, set $K_P = \prod_{i \in P} n_i$, $N_P = \text{lcm}\{n_i : i \in P\}$, and $\ell_P = \frac{K_P}{N_P}$.

Theorem 7.5. *Assume that a $\mu > 0$ is given such that*

$$\forall P \in \binom{[M]}{k} \quad \ell_P \lceil \mu N_P \rceil \leq L_P. \quad (14)$$

Then for any $0 \leq \beta \leq 1 - \mu$, $\mathbb{C}(n_1, \dots, n_M; \beta, \mu)$ is a k -dimensional transversal with the given parameters L_P . Moreover, if $\mu = \min_{P \in \binom{[M]}{k}} \frac{L_P}{K_P}$, then it is a full transversal.

Note that condition (14) easily implies that $\mu \leq \min_{P \in \binom{[M]}{k}} \frac{L_P}{K_P}$. The proof of Theorem 7.5 is also postponed to Section 10.

Corollary 7.6. *If the n_i numbers are pairwise relatively prime, then for $\mu = \min_{P \in \binom{[M]}{k}} \frac{L_P}{K_P}$ and for any $0 \leq \beta \leq 1 - \mu$, $\mathbb{C}(n_1, \dots, n_M; \beta, \mu)$ is a full k -dimensional transversal with these parameters.*

Proof. It is enough to check that (14) holds in Theorem 7.5 for this μ . Let $P \in \binom{[M]}{k}$. From the fact that the n_i numbers are relatively prime, it follows that $K_P = N_P$ and $\ell_P = 1$. Therefore $\ell_P \lceil \mu N_P \rceil \leq \lceil \frac{L_P}{N_P} \cdot N_P \rceil = L_P$. \square

Corollary 7.6 ensures that we have a full k -dimensional transversal for all $k \in [M]$ and all allowed settings of $\{L_P : P \in \binom{[M]}{k}\}$ whenever n_1, \dots, n_M are relatively prime. Unfortunately, this does not allow us to choose parameters that give MOAs, i.e. for values of L_P such that $\frac{K_P}{L_P}$ is constant. We can still use the construction in (13) to find such transversal, but we need to put more restrictions on the possible values of the L_P .

Corollary 7.7. *Assume that there is a constant $0 < \mu \leq 1$ such that for each $P \in \binom{[M]}{k}$, μN_P is an integer and $L_P = \mu K_P$. Then, for every $0 \leq \beta \leq 1 - \mu$, $\mathbb{C}(n_1, \dots, n_M; \beta, \mu)$ is a full k -dimensional transversal, and provides a simple MOA of strength $M - k$.*

Proof. The condition on μ gives $\ell_P \lceil \mu N_P \rceil = \ell_P \mu N_P = \mu K_P = L_P$ and $\mu = \min_P \frac{L_P}{K_P}$; the statement follows from Theorem 7.5. \square

While the conditions of the theorem may at first glance seem restrictive, we can easily satisfy them. For a given $k \in [M]$ we chose a sequence of integers j_1, j_2, \dots, j_M , and set $n_i = \prod_{q=1}^i j_q$. Set q as one of the divisors of n_k and $\mu = \frac{1}{q}$. It is clear that this choice of μ satisfies the conditions of Theorem 7.7, since for each $P \in \binom{[M]}{k}$ we have that $N_P = \text{lcm}\{n_i : i \in P\} = n_{\max P}$. By the choice of the n_i 's and the fact that $k \leq \max P$, n_k divides N_P . Since μn_k is an integer, so is μN_P . Thus, for each $P \in \binom{[M]}{k}$ if we chose $L_P = \mu N_P$, then the construction gives a simple MOA with the given parameters.

We also provide two "generic" constructions to create new full k -dimensional multi-transversals and MOAs from already known ones, under some numerical conditions: "linear combination", and "tensor product". The correctness of these constructions is straightforward from the definitions.

Proposition 7.8. [Linear Combination for Transversals.]

- (i) Let $j \in \mathbb{Z}^+$ and for each $\ell \in [j]$ let \mathcal{T}_ℓ be a k -dimensional multi-transversal on π_M with parameters $L_P^{(\ell)} : P \in \binom{[M]}{k}$. Assume that for all $\ell \in [j]$ positive reals α_ℓ are given such that for all $(i_1, \dots, i_M) \in \pi_M$ the quantity $\sum_{\ell=1}^j \alpha_\ell \cdot \#[(i_1, \dots, i_M), \mathcal{T}_\ell]$ is an integer, and let

$$\mathcal{T}^* = \overline{\{(i_1, \dots, i_M) : \sum_{\ell=1}^j \alpha_\ell \cdot \#[(i_1, \dots, i_M), \mathcal{T}_\ell] : (i_1, \dots, i_M) \in \pi_M\}}.$$

Then \mathcal{T}^* is a k -dimensional multi-transversal on π_M with parameters $L_P^* := \lfloor \sum_{\ell=1}^j \alpha_\ell L_P^{(\ell)} \rfloor : P \in \binom{[M]}{k}$.

- (ii) Assume further that each \mathcal{T}_ℓ above is a full multi-transversal and there is a common $A \in \binom{[M]}{k}$ on which all \mathcal{T}_ℓ simultaneously meet the bound, i.e.

$$\forall \ell \in [j] \quad L_A^{(\ell)} \prod_{j \notin A} n_j = \min_{P \in \binom{[M]}{k}} \left(L_P^{(\ell)} \prod_{j \notin P} n_j \right).$$

Then \mathcal{T}^* is a full multi-transversal as well. \square

Since the condition is true when the α_ℓ are all integers, this means in particular that if \mathcal{T}_1 and \mathcal{T}_2 are both k -dimensional multi-transversals, then so is $\mathcal{T}_1 \uplus \mathcal{T}_2$.

Proposition 7.9. [Linear Combination for MOAs.] Let $j \in \mathbb{Z}^+$ and for each $\ell \in [j]$ let \mathcal{T}_ℓ be a full k -dimensional multi-transversal on π_M with

parameters $L_P^{(\ell)} : P \in \binom{[M]}{k}$ such that $L_P^\ell \cdot \prod_{j \notin P} n_j$ is independent of P (i.e. \mathcal{T}_ℓ is a MOA). Let nonzero reals α_ℓ be given for all $\ell \in [j]$ such that for all $(i_1, \dots, i_M) \in \pi_M$ the quantity $\sum_{\ell=1}^j \alpha_\ell \cdot \#[(i_1, \dots, i_M), \mathcal{T}_\ell]$ is a non-negative integer, and let $\mathcal{T}^* = \sum_{\ell=1}^j \alpha_\ell \mathcal{T}_\ell$ be defined as

$$\mathcal{T}^* = \overline{\{(i_1, \dots, i_M) \mid \sum_{\ell=1}^j \alpha_\ell \cdot \#[(i_1, \dots, i_M), \mathcal{T}_\ell] : (i_1, \dots, i_M) \in \pi_M\}}.$$

Then \mathcal{T}^* is a full k -dimensional multi-transversal on π_M with parameters $L_P^* = \sum_{\ell=1}^j \alpha_\ell L_P^{(\ell)} : P \in \binom{[M]}{k}$, moreover, $L_P^* \prod_{j \notin P} n_j$ is independent of P (with other words, \mathcal{T}^* is a MOA). \square

In Proposition 7.8, chose $j = 2$, and MOAs \mathcal{T}_1 and \mathcal{T}_2 such that $\#[\vec{i}, \mathcal{T}_2] \geq \#[\vec{i}, \mathcal{T}_1]$ for all $\vec{i} \in \pi_M$. Then setting $\alpha_\ell = (-1)^\ell$ for $\ell \in [2]$ satisfies the conditions of Proposition 7.8 and $\mathcal{T}^* = \sum_{\ell=1}^2 \alpha_\ell \mathcal{T}_\ell = \mathcal{T}_2 \setminus \mathcal{T}_1$; this type of linear combination is exactly the relative complementation on MOAs. Accordingly, if a MOA contains another one with the same strength as a subarray, erasing the rows of the subarray results in a new MOA.

Proposition 7.8 allows us to use the construction in (13) to build simple MOAs different from the ones in (13).

Corollary 7.10. *Let n_1, \dots, n_M , and $0 < \mu < 1$ be given such that they satisfy the conditions of Corollary 7.7. For a fixed positive integer Q , and for each $i \in [2Q+1]$ let β_i be given such that $0 \leq \beta_1 < \beta_2 < \dots < \beta_{2Q+1} < \beta_1 + \mu \leq 1$ and $\beta_{2Q+1} \leq 1 - \mu$. Define $\mathcal{I} \subseteq [0, 1)$ by*

$$\mathcal{I} = \left(\bigcup_{\ell=1}^Q [\beta_{2\ell-1}, \beta_{2\ell}) \right) \cup [\beta_{2Q+1}, \beta_1 + \mu) \cup \left(\bigcup_{\ell=1}^Q [\beta_{2\ell} + \mu, \beta_{2\ell+1} + \mu) \right).$$

Then the following is a k -dimensional transversal on π_M with parameters $L_P = \mu \prod_{q \in P} n_q$ and provides a simple MOA of strength $M - k$:

$$\mathcal{T} = \left\{ (i_1, \dots, i_M) \in \pi_M : \left\langle \sum_{r=1}^M \frac{i_r}{m_r} \right\rangle \in \mathcal{I} \right\}.$$

Proof. For $\ell \in [2Q+1]$ let $\mathcal{T}_\ell = \mathbb{C}(n_1, \dots, n_M; \beta_\ell, \mu)$. By Corollary 7.7, each \mathcal{T}_ℓ is a full k -dimensional transversal on π_M with parameters $L_P = \mu \prod_{j \in P} n_j$ satisfying the conditions of Proposition 7.8. Also, using $\alpha_\ell = (-1)^{\ell+1}$ we obtain that $\mathcal{T} = \sum_{\ell=1}^{2Q+1} \alpha_\ell \mathcal{T}_\ell$. The statement follows from Proposition 7.8 and the fact that $\sum_{\ell=1}^{2Q+1} \alpha_\ell = 1$. \square

Proposition 7.11. [Tensor product.]

- (i) Assume that \mathcal{T}_1 and \mathcal{T}_2 are k -dimensional multi-transversals on $\prod_{j=1}^M [n_j^{(1)}]^\star$ and $\prod_{j=1}^M [n_j^{(2)}]^\star$ with parameters $L_P^{(1)}$ and $L_P^{(2)}$ ($P \in \binom{[M]}{k}$), respectively. Then

$$\mathcal{T} = \overline{\left\{ (a_1 n_1^{(2)} + b_1, \dots, a_M n_M^{(2)} + b_M)^{\#[(a_1, \dots, a_M), \mathcal{T}_1] \cdot \#[(b_1, \dots, b_M), \mathcal{T}_2]} \right\}}$$

is a k -dimensional multi-transversal on $\prod_{j=1}^M [n_j^{(1)} n_j^{(2)}]^\star$ with parameters $L_P = L_P^{(1)} L_P^{(2)}$.

- (ii) Assume that \mathcal{T}_1 and \mathcal{T}_2 above are full multi-transversals, and assume that there exists an $A \in \binom{[M]}{k}$, in which both meet the bound set by A , i.e. for $i \in \{1, 2\}$ we have

$$L_A^{(i)} \prod_{j \notin A} n_j^{(i)} = \min_{P \in \binom{[M]}{k}} \left(L_P^{(i)} \prod_{j \notin P} n_j^{(i)} \right).$$

Then \mathcal{T} is a full multi-transversal as well. \square

Condition (ii) holds, in particular, if (3) holds for both \mathcal{T}_1 and \mathcal{T}_2 , therefore the tensor product of MOAs of the same constraint and the same strength is a MOA of the same constraint and the same strength, using in the i^{th} column of \mathcal{T} the Cartesian product of the symbol sets of the i -th columns of \mathcal{T}_1 and \mathcal{T}_2 with appropriate multiplicities.

8 Proofs to the Sperner type results

In the proofs of this section we will frequently make use of the following structure. Let \mathcal{F} be a multi-family on $\bigsqcup_{i \in [M]} X_i$. Fix a $D \subseteq [M]$ and let $F \subseteq X \setminus \bigsqcup_{i \in D} X_i$. We define

$$\mathcal{F}(F; D) = \overline{\left\{ (E \setminus F)^{\#[E, \mathcal{F}]} : E \cap \bigsqcup_{i \in [M] \setminus D} X_i = F \right\}}.$$

The following are clear from the definitions.

Lemma 8.1. *Let \mathcal{F} be a k -dimensional M -part Sperner multi-family with parameters $L_P : P \in \binom{[M]}{k}$. Fix $k \leq N \leq M$, a $D \in \binom{[M]}{N}$ and let $F \subseteq X \setminus \bigsqcup_{i \in D} X_i$.*

(i) $\mathcal{F}(F; D)$ is a k -dimensional N -part Sperner multi-family on $\biguplus_{i \in D} X_i$ with parameters $L_P : P \in \binom{D}{k}$.

(ii) If \mathcal{F} is a simple family, so is $\mathcal{F}(F; D)$. \square

Proof to Theorem 4.1: First assume $M = k$, and call our multi-family \mathcal{F}' instead of \mathcal{F} . For each $i \in [M]$, there are $m_i!$ (simple) chains of maximum size (i.e. of length n_i) in X_i . We count the number of ordered $(k+1)$ -tuples in the following multiset in two ways:

$$\overline{\left\{ (E, \mathcal{C}_1, \dots, \mathcal{C}_k)^{\# [E, \mathcal{F}']} : E \in \prod_{i \in [M]} \mathcal{C}_i, \text{ where } \mathcal{C}_i \text{ is chain of size } n_i \text{ in } X_i \right\}}.$$

Since each chain product $\prod_{i=1}^M \mathcal{C}_i$ contains at most $L_{[M]}$ sets from \mathcal{F}' by definition, the number of such $(k+1)$ -tuples is at most $L_{[M]} \prod_{i \in [M]} m_i!$. Since each $E \in \mathcal{F}'$ can be extended to precisely $\prod_{i \in [M]} |E \cap X_i|!(m_i - |E \cap X_i|)!$ chain products with each chain being maximum size, we have that

$$\sum_{E \in \mathcal{F}'} \prod_{i \in [M]} |E \cap X_i|!(m_i - |E \cap X_i|)! \leq L_{[M]} \prod_{i \in [M]} m_i!$$

from which the claimed inequality follows in the form

$$\sum_{E \in \mathcal{F}'} \frac{1}{\prod_{i \in [M]} \binom{m_i}{|E \cap X_i|}} \leq L_{[M]}.$$

Now assume $M > k$ and take an arbitrary $P \in \binom{[M]}{k}$ to prove the theorem for our multi-family \mathcal{F} . Take an $F \subseteq X \setminus \biguplus_{i \in P} X_i$, and assume $f_i = |F \cap X_i|$ for $i \notin P$. By Lemma 8.1, $\mathcal{F}(F; P)$ is a k -dimensional k -part Sperner multi-family with parameter L_P , and therefore, using Theorem 4.1 we get

$$\sum_{E \in \mathcal{F}(F; P)} \frac{1}{\prod_{i \in P} \binom{m_i}{|E \cap X_i|}} \leq L_P. \quad (15)$$

From this we can write for any fixed sequence f_i ($i \notin P$):

$$\sum_{\substack{F: F \subseteq X \setminus \biguplus_{i \in P} X_i \\ |F \cap X_i| = f_i, i \notin P}} \sum_{E \in \mathcal{F}(F; P)} \frac{1}{\prod_{i \in P} \binom{m_i}{|E \cap X_i|} \prod_{i \notin P} \binom{m_i}{f_i}} \leq L_P.$$

Finally, summing up the previous inequality for $f_i = 0, 1, \dots, m_i$, for all $i \notin P$, we obtain the theorem. \square

To prove Theorem 4.4, we first need the following definitions: Let $\mathcal{F} \subseteq X$ be an non-empty M -part multi-family, and let $j \in [M]$. We define $\text{high}_j(\mathcal{F})$ and $\text{low}_j(\mathcal{F})$ as the largest and smallest levels in X_j that the trace \mathcal{F}_{X_j} in X_j intersects. With other words,

$$\begin{aligned}\text{high}_j(\mathcal{F}) &= \max \left\{ q \in [n_j]^* : \mathcal{F}_{X_j} \cap \binom{X_j}{q} \neq \emptyset \right\}, \\ \text{low}_j(\mathcal{F}) &= \min \left\{ q \in [n_j]^* : \mathcal{F}_{X_j} \cap \binom{X_j}{q} \neq \emptyset \right\}.\end{aligned}$$

First, we will need the following:

Lemma 8.2. *Let $M > 1$, $j \in [M - 1]$ and let \mathcal{F}' be an M -dimensional M -part Sperner family with $\text{high}_j(\mathcal{F}') > \text{low}_j(\mathcal{F}')$ that satisfies (5), and let $E_0 \in \mathcal{F}'_{X_M}$ be fixed. Then there is an M -dimensional M -part Sperner family \mathcal{F} that also satisfies (5) such that for all $i \in [M] \setminus \{j\}$ we have $\mathcal{F}_{X_i} \subseteq \mathcal{F}'_{X_i}$, $E_0 \in \mathcal{F}_{X_M}$ and $\text{high}_j(\mathcal{F}) - \text{low}_j(\mathcal{F}) = \text{high}_j(\mathcal{F}') - \text{low}_j(\mathcal{F}') - 1$.*

Proof. Let $t = \text{high}_j(\mathcal{F}')$ and $\mathcal{B} = \{B_1, \dots, B_s\} = \mathcal{F}'_{X_j} \cap \binom{X_j}{t}$. For $i \in [s]$ let $\mathcal{E}_i = \{E \in \mathcal{F}' : E \cap X_j = B_i\}$ and $\mathcal{E} = \cup_{i=1}^s \mathcal{E}_i$. Given $A \subseteq X_j$, we define

$$w(A) = \sum_{E \in \mathcal{F}' : E \cap X_j = A} \frac{1}{\prod_{i: i \neq j} \binom{m_i}{|E \cap X_i|}}.$$

We also assume (w.l.o.g.) that $w(B_1) \geq \dots \geq w(B_s)$; we will use $w_i := w(B_i)$. Using this notation we can rewrite (5) as

$$\sum_{A \subseteq X_j} \frac{w(A)}{\binom{m_j}{|A|}} = 1,$$

or equivalently

$$\sum_{i=1}^s \frac{w_i}{\binom{m_j}{t}} + \delta = 1; \quad \delta := \sum_{A \in \mathcal{F}'_{X_j} \setminus \mathcal{B}} \frac{w(A)}{\binom{m_j}{|A|}}. \quad (16)$$

Recall the following well-known fact (see e.g. [7]) that for every $t \in [n]$ and a subset $\mathcal{A} \subseteq \binom{[n]}{t}$ we have

$$\frac{|\partial(\mathcal{A})|}{\binom{n}{t-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{t}}, \quad (17)$$

where $\partial(\mathcal{A})$, called the *lower shadow* of \mathcal{A} , is defined as $\partial(\mathcal{A}) = \{E \in \binom{[n]}{t-1} : E \subsetneq F \text{ for some } F \in \mathcal{A}\}$. Moreover, equality in (17) holds if and only if $\mathcal{A} = \binom{[n]}{t}$.

Similar inequality holds for the *upper shadow* $\mathfrak{G}(\mathcal{A})$ of \mathcal{A} defined as $\mathfrak{G}(\mathcal{A}) = \{E \in \binom{[n]}{t+1} : E \supsetneq F \text{ for some } F \in \mathcal{A}\}$, that is $|\mathfrak{G}(\mathcal{A})|/\binom{n}{t+1} \geq |\mathcal{A}|/\binom{n}{t}$ (with equality if and only if $\mathcal{A} = \binom{[n]}{t}$).

Let us denote $\mathbb{B}_i = \{B_1, \dots, B_i\}$; $i = 1, \dots, s$ (thus $\mathbb{B}_i \subsetneq \mathbb{B}_{i+1}$ and $\mathbb{B}_s = \mathcal{B}$). We define then the following partition of $\partial(\mathcal{B}) = \mathbb{B}'_1 \cup \dots \cup \mathbb{B}'_s$:

$$\mathbb{B}'_1 = \partial(\mathbb{B}_1), \quad \mathbb{B}'_i = \partial(\mathbb{B}_i) \setminus \partial(\mathbb{B}_{i-1}); \quad i = 2, \dots, s.$$

Then, in view of (17), we have

$$\frac{\sum_{\ell=1}^i |\mathbb{B}'_\ell|}{\binom{m_j}{t-1}} = \frac{|\partial(\mathbb{B}_i)|}{\binom{m_j}{t-1}} \geq \frac{|\mathbb{B}_i|}{\binom{m_j}{t}} = \frac{i}{\binom{m_j}{t}}; \quad i = 1, \dots, s, \quad (18)$$

with strict inequality if $s < \binom{m_j}{t}$.

Recall that \mathcal{A} is an M -dimensional M -part Sperner family with parameter $L_{[M]} = 1$ precisely when for all $A, B \in \mathcal{A}$ with $A \neq B$, there is an $\ell \in [M]$ such that $A \cap X_\ell$ and $B \cap X_\ell$ are incomparable by the subset relation.

For ease of description, let us represent each family \mathcal{E}_i , defined above, by the direct product $\mathcal{E}_i = \{B_i\} \times \mathcal{H}_i$, where $\mathcal{H}_i = \mathcal{F}'(B_i; [M] \setminus \{j\})$ is an $(M-1)$ -dimensional $(M-1)$ -part Sperner family in the partition set $\bigsqcup_{i \in [M] \setminus \{j\}} X_i$.

We now construct a new family \mathcal{E}^* from \mathcal{E} as follows. We replace each \mathcal{E}_i by $\mathcal{E}_i^* := \mathbb{B}'_i \times \mathcal{H}_i$; $i = 1, \dots, s$ and define $\mathcal{E}^* = \cup_{i=1}^s \mathcal{E}_i^*$. Observe now that for each $A^* \in \mathcal{E}_i^*$ there is an $A \in \mathcal{E}_i$ such that $A^* \cap X_\ell = A \cap X_\ell$ for all $\ell \in [M] \setminus \{j\}$ and $A^* \cap X_j \subsetneq A \cap X_j$. This implies that $\mathcal{E}^* \cap \mathcal{F}' = \emptyset$, since \mathcal{F}' is an M -dimensional M -part Sperner family with parameter $L_{[M]} = 1$.

Moreover, it is not hard to see that $\mathcal{F}^* := (\mathcal{F}' \setminus \mathcal{E}) \cup \mathcal{E}^*$ is an M -dimensional M -part Sperner family with parameter $L_{[M]}^* = 1$. If we have that A, B are different elements of $\mathcal{F}' \setminus \mathcal{E}$, then the required property follows from the fact that A, B are both elements of \mathcal{F}' . If A^*, B^* are different elements of \mathcal{E}^* , then either $A^* \cap X_j$ and $B^* \cap X_j$ are both incomparable, or $A^*, B^* \in \mathcal{E}_i$ for some i , in which case the corresponding sets $A, B \in \mathcal{E}_i \subseteq \mathcal{F}'$ agree with A^*, B^* on $X \setminus X_j$ and $A^* \cap X_j = B^* \cap X_j = B_i$, from which the required property follows. Finally, take $A^* \in \mathcal{E}_i^*$ for some i and $B \in \mathcal{F}' \setminus \mathcal{E}$, and let $A \in \mathcal{E}_i \subseteq \mathcal{F}'$ be the corresponding set. If $A \cap X_j$ and $B \cap X_j$ are comparable, then from the fact that t was the largest level of \mathcal{F}'_{X_j} we get that $B \cap X_j \subseteq A^* \cap X_j \subsetneq B_i = A \cap X_j$; and from the fact that A, B are both elements of \mathcal{F}' and $A \setminus X_j = A^* \setminus X_j$ the required property follows.

Therefore \mathcal{F}^* is an M -dimensional M -part Sperner family with parameter 1. Thus, for \mathcal{F}^* the following inequality must hold:

$$\sum_{E \in \mathcal{F}^*} \frac{1}{\prod_{i=1}^M \binom{m_i}{|E \cap X_i|}} = \sum_{i=1}^s \frac{|\mathbb{B}'_i| \cdot w_i}{\binom{m_j}{t-1}} + \delta \leq 1. \quad (19)$$

On the other hand, (18) together with $w_1 \geq \dots \geq w_s \geq 0 =: w_{s+1}$ implies that

$$\sum_{\ell=1}^s \frac{|\mathbb{B}'_\ell| \cdot w_\ell}{\binom{m_j}{t-1}} = \sum_{i=1}^s \sum_{\ell=1}^i \frac{|\mathbb{B}'_\ell| \cdot (w_i - w_{i+1})}{\binom{m_j}{t-1}} \geq \sum_{i=1}^s \frac{i \cdot (w_i - w_{i+1})}{\binom{m_j}{t}} = \sum_{i=1}^s \frac{w_i}{\binom{m_j}{t}}. \quad (20)$$

In fact, the latter means that $\mathcal{B} = \binom{X_j}{t}$, otherwise we have strict inequality in (20) a contradiction with (19), in view of (16). Thus, for the new family \mathcal{F}^* we have

$$\sum_{E \in \mathcal{F}^*} \frac{1}{\prod_{i=1}^M \binom{m_i}{|E \cap X_i|}} = 1.$$

Moreover, $\text{high}_j(\mathcal{F}^*) = \text{high}_j(\mathcal{F}') - 1$ and $\text{low}_j(\mathcal{F}^*) = \text{low}_j(\mathcal{F}')$, so $\text{high}_j(\mathcal{F}^*) - \text{low}_j(\mathcal{F}^*) = \text{high}_j(\mathcal{F}') - \text{low}_j(\mathcal{F}') - 1$. In addition, for all $\ell \in [M] \setminus \{j\}$ we have $\mathcal{E}_{X_\ell}^* \subseteq \mathcal{E}_{X_\ell}$, therefore $\mathcal{F}_{X_\ell}^* \subseteq \mathcal{F}'_{X_\ell}$. Therefore, if $E_0 \in (\mathcal{F}' \setminus \mathcal{E})_{X_M}$, i.e. the trace of $\mathcal{F}' \setminus \mathcal{E}$ in X_M contains E_0 , then setting $\mathcal{F} := \mathcal{F}^*$ will give the required family.

If $E_0 \notin (\mathcal{F}' \setminus \mathcal{E})_{X_M}$, then, since $\mathcal{F}'_{X_M} \setminus \mathcal{E}_{X_M} \subseteq (\mathcal{F}' \setminus \mathcal{E})_{X_M}$ and $E_0 \in \mathcal{F}'_{X_M}$ we must have that $E_0 \in \mathcal{E}_{X_M}$. Similar to the described "pushing down" transformation in \mathcal{F}' we can apply "pushing up" transformation with respect to the smallest level \mathcal{D} in \mathcal{F}'_{X_j} , replacing it by its upper shadow $\mathfrak{G}(\mathcal{D})$ to obtain the new family \mathcal{F} . Since $\mathcal{D} \neq \mathcal{B}$, we now have $\mathcal{E} \subseteq \mathcal{F}$, therefore $E_0 \in \mathcal{F}_{X_M}$. All other required conditions follow as before. \square

Proof to Theorem 4.4: Let \mathcal{F}' be an M -dimensional M -part Sperner family with parameter 1 satisfying (5). Without loss of generality assume, contrary to the statement of the theorem, that the trace \mathcal{F}'_{X_M} of \mathcal{F}' in X_M contains an incomplete level, i.e. there is a $y_M \in [n_M]^*$ such that for $\mathcal{G} = \mathcal{F}' \cap \binom{X_M}{y_M}$ we have that $\emptyset \subsetneq \mathcal{G} \subsetneq \binom{X_M}{y_M}$. Fix an $E_0 \in \mathcal{G}$.

Let $\mathcal{F}^{(0)} := \mathcal{F}'$. We will define a sequence $\mathcal{F}^{(1)}, \dots, \mathcal{F}^{(M-1)}$ of M -dimensional M -part Sperner families such that for each $\ell \in [M-1]$ the following hold:

(i) Equality (5) holds for $\mathcal{F}^{(\ell)}$, with other words

$$\sum_{E \in \mathcal{F}^{(\ell)}} \frac{1}{\prod_{i=1}^M \binom{m_i}{|E \cap X_i|}} = 1. \quad (21)$$

(ii) There is a $y_\ell \in [n_\ell]^\star$ such that $\mathcal{F}_{X_\ell}^{(\ell)} \subseteq \binom{X_\ell}{y_\ell}$, with other words the trace of $\mathcal{F}^{(\ell)}$ in X_ℓ consist of a single (not necessarily full) level.

(iii) For each $i \in [M] \setminus \{\ell\}$, $\mathcal{F}_{X_i}^{(\ell)} \subseteq \mathcal{F}_{X_i}^{(\ell-1)}$.

(iv) $E_0 \in \mathcal{F}_{X_M}^{(\ell)}$.

Once this sequence is defined, it follows that for all $j \in [M-1]$ we have that $\mathcal{F}_{X_j}^{(M-1)} \subseteq \binom{X_j}{y_j}$, also $E_0 \in \left(\mathcal{F}_{X_M}^{(M-1)} \cap \binom{X_M}{y_M}\right) \subseteq \mathcal{G} \subsetneq \binom{X_M}{y_M}$, therefore the trace of $\mathcal{F}^{(M-1)}$ in X_M contains an incomplete level.

Also, for all $F \in X \setminus X_M$ we must have that $\mathcal{F}^{(M-1)}(F; \{M\})$ is a 1-dimensional 1-part Sperner family with parameter 1, therefore it satisfies (6) with the parameter set to 1. In view of these facts, using (21) for $\ell = M-1$ we get that

$$\begin{aligned} 1 &= \sum_{E \in \mathcal{F}^{(M-1)}} \frac{1}{\prod_{i=1}^M \binom{m_i}{|E \cap X_i|}} \\ &= \frac{1}{\prod_{i=1}^{M-1} \binom{m_i}{t_i}} \sum_{F \subseteq X \setminus X_M} \left(\sum_{E \in \mathcal{F}^{(M-1)}(F; \{M\})} \frac{1}{\binom{m_M}{|E \cap X_M|}} \right) \\ &\leq \frac{1}{\prod_{i=1}^{M-1} \binom{m_i}{t_i}} \sum_{F \subseteq X \setminus X_M} 1 = 1. \end{aligned}$$

This implies that for all $F \subseteq X \setminus X_M$, (6) holds with equality for $\mathcal{F}^{(M-1)}(F, \{M\})$, so by Lemma 4.5 we get that $\mathcal{F}^{(M-1)}(F, \{M\})$ is a full level. Since

$$\mathcal{F}_{X_M}^{(M-1)} = \bigcup_{F \subseteq X \setminus X_M} \mathcal{F}^{(M-1)}(F, \{M\}),$$

this implies that $\mathcal{F}_{X_M}^{(M-1)}$ must consist of full levels only, a contradiction.

Note that $\mathcal{F}^{(0)}$ is defined, it satisfies (5), and it does not need to satisfy any other conditions. All that remains to show is that $\mathcal{F}^{(\ell)}$ can be defined for each $\ell \in [M-1]$ such that it satisfies the conditions (i)–(iv).

To this end, assume that $j \in [M-1]$ and $\mathcal{F}^{(j-1)}$ is already given satisfying all required conditions. Let $Q = \text{high}_j(\mathcal{F}^{(j-1)}) - \text{low}_j(\mathcal{F}^{(j-1)})$. If $Q = 0$, then $\mathcal{F}_{X_j}^{(j-1)}$ consists of a single, not necessarily full, level, and we set $\mathcal{F}^{(j)} = \mathcal{F}^{(j-1)}$; (i)–(iv) are clearly satisfied.

If $Q > 0$, then let $\mathcal{K}^{(0)} = \mathcal{F}^{(j-1)}$. By Lemma 8.2 we can define a sequence $\mathcal{K}^{(1)}, \dots, \mathcal{K}^{(Q)}$ of M -dimensional M -part Sperner families with parameter 1 such that for all $\ell \in [Q]$ the following hold:

- (a) $\mathcal{K}^{(\ell)}$ satisfies (5).
- (b) For all $i \in [M] \setminus \{j\}$ we have $\mathcal{K}_{X_i}^{(\ell)} \subseteq \mathcal{K}_{X_i}^{(\ell-1)}$.
- (c) $E_0 \in \mathcal{K}_{X_M}^{(\ell)}$.
- (d) $\text{high}_j(\mathcal{K}^{(\ell)}) - \text{low}_j(\mathcal{K}^{(\ell)}) = \text{high}_j(\mathcal{K}^{(\ell-1)}) - \text{low}_j(\mathcal{K}^{(\ell-1)}) - 1$.

It follows that $\text{high}_j(\mathcal{K}^{(Q)}) = \text{low}_j(\mathcal{K}^{(Q)})$ and we set $\mathcal{F}^{(j)} = \mathcal{K}^{(Q)}$; (i)–(iv) are clearly satisfied. \square

It only remains to prove Theorem 4.2. We will start with a series of lemmata. The first lemma states for multi-families what Theorem 6.2 in [1] stated for simple families:

Lemma 8.3. *Let $1 \leq M$ and \mathcal{F} be a 1-dimensional M -part Sperner multi-family with parameters $L_{\{i\}}$ for $i \in [M]$ satisfying (4) with equalities, i.e.*

$$\forall i \in [M] \quad \sum_{(i_1, \dots, i_M) \in \pi_M} \frac{p_{i_1, \dots, i_M}}{\prod_{j=1}^M \binom{m_j}{i_j}} = \frac{L_{\{i\}}}{n_i} \prod_{j=1}^M n_j. \quad (22)$$

Then \mathcal{F} is homogeneous.

Proof. For $M = 1$ the statement is proved in Lemma 4.5. Let $M \geq 2$ and take an arbitrary $F \in \mathcal{F}$. We set $F_i = F \cap X_i$ and $G_i = F \setminus F_i$. By Lemma 8.1 that for each $j \in [M]$, $\mathcal{F}(G_j; \{j\})$ is a (1-dimensional 1-part) Sperner multi-family with parameter $L_{\{j\}}$. From the proof of Theorem 4.1 and (22) we get that equality must hold in (15), i.e.

$$\sum_{E \in \mathcal{F}(G_j; \{j\})} \frac{1}{\binom{m_j}{|E \cap X_i|}} = L_{\{j\}},$$

which by Lemma 4.5 implies that $\mathcal{F}(G_j; \{j\})$ is homogeneous. In particular this means that for all $A \in \mathcal{F}$ for all $j \in [M]$ if B is a set such that $|A \cap X_j| = |B \cap X_j|$ and for all $i \in [M] \setminus \{j\}$ we have $A \cap X_i = B \cap X_i$, then the $\#[A, \mathcal{F}] = \#[B, \mathcal{F}]$. If A, B are sets with the same profile vector, we define the sequence $A = Y_0, Y_1, \dots, Y_M = B$ by $Y_i = (Y_{i-1} \setminus X_i) \uplus (B \cap X_i)$ for all $i \in [M]$. It follows that $\#[Y_{i-1}, \mathcal{F}] = \#[Y_i, \mathcal{F}]$, and so $\#[A, \mathcal{F}] = \#[B, \mathcal{F}]$. Thus \mathcal{F} is homogeneous. \square

Lemma 8.4. *Let $1 \leq k$ and let \mathcal{F} be a k -dimensional $(k+1)$ -part Sperner multi-family with parameters $L_{[k+1] \setminus \{i\}}$ for $i \in [k+1]$ satisfying (4) with equality, i.e.*

$$\forall i \in [k+1] \quad \sum_{(i_1, \dots, i_{k+1}) \in \pi_{k+1}} \frac{p_{i_1, \dots, i_{k+1}}}{\prod_{j=1}^{k+1} \binom{m_j}{i_j}} = L_{[k+1] \setminus \{i\}} n_i. \quad (23)$$

Then \mathcal{F} is homogeneous.

Proof. The proof is induction on k . For $k = 1$, it is proved in Lemma 8.3. By Lemma 8.1 we have that for each $j \in [M]$ and each $F \subseteq X_j$, $\mathcal{F}(F; [k+1] \setminus \{j\})$ is a k -dimensional k -part Sperner multi-family with parameter $L_{[k+1] \setminus \{j\}}$. From the proof of Theorem 4.1 and (23) we get that equality must hold in (15), i.e.

$$\sum_{E \in \mathcal{F}(F; [k+1] \setminus \{j\})} \frac{1}{\prod_{i: i \neq j} \binom{m_i}{|E \cap X_i|}} = L_{[k+1] \setminus \{j\}}. \quad (24)$$

Fixing a maximal chain $F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_{m_j}$ in X_j , we get that $\mathcal{F}' = \biguplus_{q=0}^{m_j} \mathcal{F}(F_q; [k+1] \setminus \{j\})$ is a $(k-1)$ -dimensional k -part Sperner multi-family with parameters $L'_{[k+1] \setminus \{j, \ell\}} := L_{[k+1] \setminus \{\ell\}} : \ell \in [k+1] \setminus \{j\}$, moreover, using (24) for each $F = F_q$ we get that

$$\sum_{E \in \mathcal{F}'} \frac{1}{\prod_{i: i \neq j} \binom{m_i}{|E \cap X_i|}} = \sum_{q=0}^{m_j} \left(\sum_{E \in \mathcal{F}(F_q; [k+1] \setminus \{j\})} \frac{1}{\prod_{i: i \neq j} \binom{m_i}{|E \cap X_i|}} \right) = n_j L_{[k+1] \setminus \{j\}}.$$

By (23) we have that $L_{[k+1] \setminus \{j\}} n_j = L_{[k+1] \setminus \{\ell\}} n_\ell = L'_{[k+1] \setminus \{j, \ell\}} n_\ell$, therefore \mathcal{F}' is homogeneous by the induction hypothesis. In particular this means that for all $A \in \mathcal{F}$ for all $j \in [M]$ if B is a set such that $|A \cap X_j| = |B \cap X_j|$ and for all $i \in [M] \setminus \{j\}$ we have $A \cap X_i = B \cap X_i$, then the $\#[A, \mathcal{F}] = \#[B, \mathcal{F}]$. This implies, as in the proof of Lemma 8.3, that \mathcal{F} is homogeneous. \square

Lemma 8.5. *Let $2 \leq k \leq M - 1$ and let \mathcal{F} be a k -dimensional M -part Sperner multi-family with parameters L_P for $P \in \binom{[M]}{k}$ satisfying (4) with equalities. Then \mathcal{F} is homogeneous.*

Proof. The proof is essentially the same as the proof of Lemma 8.4. If $M = k + 1$, we are done by Lemma 8.4. If $M > k + 1$, by Lemma 8.1 we get that for each $D \in \binom{[M]}{k+1}$ and $F \subseteq X \setminus \biguplus_{i \in D} X_i$, $\mathcal{F}(F; D)$ is a k -dimensional $(k + 1)$ -part Sperner multi-family with parameters $L_P : P \in \binom{D}{k}$. Fix an $F \subseteq X \setminus \biguplus_{i \in D} X_i$, and set $\mathcal{F}' = \mathcal{F}(F; D)$. For any $j \in D$ and $G \subseteq X_j$ we have that $\mathcal{F}'(G; D \setminus \{j\}) = \mathcal{F}(F \uplus G; D \setminus \{j\})$ and $\mathcal{F}'(G; D \setminus \{j\})$ is a k -dimensional k -part Sperner family with parameter $L_{D \setminus \{j\}}$. From the proof of Theorem 4.1 and the fact that in \mathcal{F} (4) holds with equality we get that equality must hold for all $j \in D$ and all $G \subseteq X_j$ for $\mathcal{F}'(G; D \setminus \{j\})$ in (15), i.e.

$$\sum_{E \in \mathcal{F}'(G; D \setminus \{j\})} \frac{1}{\prod_{\ell \in [M] \setminus (D \cup \{j\})} \binom{m_\ell}{|E \cap X_\ell|}} = L_{D \setminus \{j\}}. \quad (25)$$

Fixing a maximal chain $G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_{m_j}$ in X_j we get that $\mathcal{F}^* = \biguplus_{i=0}^{m_j} \mathcal{F}'(G_i; D \setminus \{j\})$ is a $(k - 1)$ -dimensional k -part Sperner multi-family with parameters $L_{P^*}^* := L_{P^* \cup \{j\}} : P^* \in \binom{D \setminus \{j\}}{k-1}$, moreover, using (25) for each G_i we get that

$$\sum_{E \in \mathcal{F}^*} \frac{1}{\prod_{\ell \in D \setminus \{j\}} \binom{m_\ell}{|E \cap X_\ell|}} = \sum_{i=1}^{m_j} \left(\sum_{E \in \mathcal{F}(G_i; D \setminus \{j\})} \frac{1}{\prod_{\ell \in D \setminus \{j\}} \binom{m_\ell}{|E \cap X_\ell|}} \right) = L_{D \setminus \{j\}} n_j.$$

Fix any $P^* \in \binom{D \setminus \{j\}}{k-1}$. Then $P^* = D \setminus \{i, j\}$ for some $i \in D \setminus \{j\}$, and from the conditions of the theorem we get that

$$L_{P^*}^* n_i = L_{P^* \cup \{j\}} n_i = L_{D \setminus \{i\}} n_i = L_{D \setminus \{j\}} n_j,$$

therefore \mathcal{F}^* is homogeneous by the induction hypothesis. This means that if $A \in \mathcal{F}$ and B is a set with the same profile vector as A , and $A \cap X_i = B \cap X_i$ for at least $M - k - 1 \geq 1$ values of i , then $\# [A, \mathcal{F}] = \# [B, \mathcal{F}]$. As before, we get that \mathcal{F} is homogeneous. \square

Proof to Theorem 4.2: Lemmata 8.3, 8.4 and 8.5 together proves part (i), and, as remarked earlier, part (ii) follows from the conditions.

(iii): By part (i), equality in (4) implies homogeneity, i.e. that for any $(i_1, \dots, i_M) \in \pi_M$ there is a positive integer r_{i_1, \dots, i_M} such that every set in \mathcal{F} that has profile vector (i_1, \dots, i_M) appears with multiplicity r_{i_1, \dots, i_M} , and

also equality in (15). Equality in (15) means that for any chain product $\mathcal{C} := \prod_{i=1}^M \mathcal{C}_i$ where \mathcal{C}_i is a maximal chain in X_i , any given $P \in \binom{[M]}{k}$ and any subset $F \subseteq X \setminus \cup_{i \in P} X_i$, each subproduct $\prod_{j \in P} \mathcal{C}_j$ of maximal chains is covered exactly L_P times by the elements of $\mathcal{F}(F; P)$, that is

$$\left| \overline{\left\{ E^{\# [E, \mathcal{F}(F; P)]} : E \in \prod_{j \in P} \mathcal{C}_j \right\}} \right| = L_P. \quad (26)$$

For a given chain product $\mathcal{C} = \prod_{i \in [M]} \mathcal{C}_i$ of maximum-size chains \mathcal{C}_i in X_i , we define

$$\mathcal{F}[\mathcal{C}] = \overline{\{ F^{\# [F, \mathcal{F}]} : F \in \mathcal{C} \}}.$$

Each $F \in \mathcal{F}[\mathcal{C}]$ is uniquely determined from its profile vector (f_1, \dots, f_M) . Let $\mathcal{T}_{\mathcal{C}}$ denote the multiset of all profile vectors of the sets in $\mathcal{F}[\mathcal{C}]$, where each profile vector appears with the multiplicity of its corresponding set in $\mathcal{F}[\mathcal{C}]$. Since \mathcal{F} is homogeneous, $\mathcal{T}_{\mathcal{C}}$ does not depend on the choice of \mathcal{C} .

We can describe now property (26) of $\mathcal{F}[\mathcal{C}]$ in terms of its profile vectors as follows: for each subset $\{i_1 < \dots < i_{M-k}\} \in \binom{[M]}{M-k}$, and each $(M-k)$ -tuple of coordinate values $(f_{i_1}, \dots, f_{i_{M-k}}) \in \prod_{j=1}^{M-k} [n_{i_j}]^*$ the set of vectors in $\mathcal{T}_{\mathcal{C}}$ where the i_j -th coordinate is f_{i_j} for $j \in [M-k]$ has size $L_{M \setminus \{i_1, \dots, i_{M-k}\}}$. Let \mathcal{T} denote the transversal corresponding to the homogeneous multi-family \mathcal{F} . Then clearly $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$ for every product of maximal chains $\mathcal{C} = \prod_{i=1}^M \mathcal{C}_i$.

We infer now that the k -dimensional multi-transversal \mathcal{T} is a simple MOA with symbol sets $S_i = \{0, 1, 2, \dots, m_i\}$, of constraint M , strength $M-k$, and index set $\mathbb{L} = \{L_P : P \in \binom{[M]}{k}\}$, with $\lambda(j_1, \dots, j_{M-k}) = L_{[M] \setminus \{j_1, \dots, j_{M-k}\}}$. This completes the proof of part (iii).

It is also clear that any MOA with the parameters described above is a k -dimensional multi-transversal corresponding to a homogeneous k -dimensional M -part Sperner multi-family \mathcal{F} with parameters $\{L_P : P \in \binom{[M]}{k}\}$ on a partitioned $(m_1 + \dots + m_M)$ -element underlying set, where the multiplicity of each element in $F \in \mathcal{F}$ is the same as the multiplicity of its profile vector (f_1, \dots, f_M) in the multi-transversal, which satisfies equality in (4). \square

9 Proofs to convex hull results

Proof to Lemma 5.1: will suffice to show that for every multi-family $\mathcal{H} \in \mathbb{A}$, there are non-negative coefficients $\lambda(I)$ for every $I \in \pi_M$ with $T(I) \in$

$\mu(\mathbb{A}(\mathfrak{L}))$, such that $\sum_I \lambda(I) = 1$ and

$$\sum_I \lambda(I) S(I) = \mathbb{P}(\mathcal{H}). \quad (27)$$

To this end, fix an $\mathcal{H} \in \mathbb{A}$ and for all $H \subseteq X$ let $\mathcal{H}_H = \overline{\{H^{\# [H, \mathcal{H}]}\}}$, with other words \mathcal{H}_H has H with the same multiplicity as \mathcal{H} , and it has no other elements. Consider the sum

$$\sum_{(\mathfrak{L}, H)} \frac{\mathbb{P}(\mathcal{H}_H)}{\prod_{j=1}^M (m_j!)} \quad (28)$$

for all ordered pairs (\mathfrak{L}, H) , where \mathfrak{L} is a product-permutation, $H \subseteq X$, and H is initial with respect to the product-permutation \mathfrak{L} . We evaluate (28) in two ways. The first way is:

$$\begin{aligned} \sum_{(\mathfrak{L}, H)} \frac{\mathbb{P}(\mathcal{H}_H)}{\prod_{j=1}^M (m_j!)} &= \sum_{\mathfrak{L}} \frac{1}{\prod_{j=1}^M (m_j!)} \left(\sum_{\substack{H \subseteq X: \\ H \text{ is initial for } \mathfrak{L}}} \mathbb{P}(\mathcal{H}_H) \right) \\ &= \sum_{\mathfrak{L}} \frac{\mathbb{P}(\mathcal{H}(\mathfrak{L}))}{\prod_{j=1}^M (m_j!)}. \end{aligned} \quad (29)$$

Observe that $\mathbb{P}(\mathcal{H}(\mathfrak{L})) \in \mu(\mathbb{A}(\mathfrak{L}))$, and therefore for every \mathfrak{L} there is a unique I such that $T(I) = \mathbb{P}(\mathcal{H}(\mathfrak{L}))$. Collecting the identical terms in the right side of (29),

$$\sum_{\mathfrak{L}} \frac{\mathbb{P}(\mathcal{H}(\mathfrak{L}))}{\prod_{j=1}^M (m_j!)} = \sum_{T(I) \in \mu(\mathbb{A}(\mathfrak{L}))} \lambda(I) T(I), \quad (30)$$

where $\lambda(I)$ is the proportion of the $\prod_{j=1}^M (m_j!)$ product-permutations such that $\mathbb{P}(\mathcal{H}(\mathfrak{L})) = T(I)$, thus $\sum_{T(I) \in \mu(\mathbb{A}(\mathfrak{L}))} \lambda(I) = 1$. Consider a fixed set H with profile vector (i_1, i_2, \dots, i_M) . There are exactly $\prod_{j=1}^M (i_j! \cdot (m_j - i_j!))$

product-chains to which H is initial. Using this, we also get:

$$\begin{aligned} \sum_{(\mathfrak{L}, H)} \frac{\mathbb{P}(\mathcal{H}_H)}{\prod_{j=1}^M (m_j!)} &= \sum_{H: H \subseteq X} \sum_{\substack{\mathfrak{L}: \\ H \text{ is initial for } \mathfrak{L}}} \frac{\mathbb{P}(\mathcal{H}_H)}{\prod_{j=1}^M (m_j!)} \\ &= \sum_{H: H \subseteq X} \frac{\prod_{j=1}^M (i_j! \cdot (m_j - i_j!))}{\prod_{j=1}^M (m_j!)} \cdot \mathbb{P}(\mathcal{H}_H) \end{aligned} \quad (31)$$

$$= \sum_{H: H \subseteq X} \frac{\mathbb{P}(\mathcal{H}_H)}{\prod_{j=1}^M \binom{m_j}{i_j}} = \left(\frac{p_{i_1, \dots, i_M}(\mathcal{H})}{\binom{m_1}{i_1} \dots \binom{m_M}{i_M}} \right)_{(i_1, \dots, i_M) \in \pi_M} \quad (32)$$

Combining (28), (29), (30), (31), and (32), we obtain

$$\left(\frac{p_{i_1, \dots, i_M}(\mathcal{H})}{\binom{m_1}{i_1} \dots \binom{m_M}{i_M}} \right)_{(i_1, \dots, i_M) \in \pi_M} = \sum_{T(I) \in \mu(\mathbb{A}(\mathfrak{L}))} \lambda(I) T(I),$$

which implies for all $(i_1, \dots, i_M) \in \pi_M$ that

$$p_{i_1, \dots, i_M}(\mathcal{H}) = \sum_{T(I) \in \mu(\mathbb{A}(\mathfrak{L}))} \left(\lambda(I) \left(\prod_{j=1}^M \binom{m_j}{i_j} \right) t_{i_1, \dots, i_M}(I) \right).$$

This proves (27). \square

Proof to Theorem 5.3: First observe that $\mu(\mathbb{A}(\mathfrak{L}))$ does not depend on \mathfrak{L} , so (9) holds. Next we have to show (10), i.e. we have to show that if $T(I) \in \mu(\mathbb{A}(\mathfrak{L}))$ for some $I \in \pi_M$ and all product-permutation \mathfrak{L} , then $S(I) \in \mu(\mathbb{A})$.

Assume $I \in \pi_M$ and $T(I) \in \mu(\mathbb{A}(\mathfrak{L}))$ for all product-permutation \mathfrak{L} . Then for each product-permutation \mathfrak{L} there is an $\mathcal{H}_{\mathfrak{L}} \in \mathbb{A}$ such that $T(I) = \mathbb{P}(\mathcal{H}_{\mathfrak{L}}(\mathfrak{L}))$. Since $\mathcal{H}_{\mathfrak{L}}$, and therefore $\mathcal{H}_{\mathfrak{L}}(\mathfrak{L})$ as well, satisfies \mathbb{M}_{Γ} , we must have that I satisfies \mathbb{M}_{Γ} . Let $\mathcal{F}_{S(I)}$ be the homogeneous multi-family that realizes the profile matrix $S(I)$, then for all $(i_1, \dots, i_M) \in \pi_M$ we have $\max\{\#[F, \mathcal{F}_{S(I)}] : \forall j \ |F \cap X_j| = i_j\} = \#[(i_1, \dots, i_M), I]$, consequently, $\mathcal{F}_{S(I)}$ satisfies \mathbb{M}_{Γ} . Thus $S(I) \notin \mu(\mathbb{A})$ implies that the homogeneous multi-family $\mathcal{F}_{S(I)}$ is not a k -dimensional M -part multi-family with parameters $L_P : P \in$

$\binom{[M]}{k}$). This means that there is a $P_0 \in \binom{[M]}{k}$, sets D_i for all $i \notin P_0$ and chains \mathcal{C}_j for all $j \in P_0$ such that

$$\left| \overline{\left\{ F^{\# [F, \mathcal{F}_{S(I)}} : \left(F \cap \biguplus_{j \in P_0} X_j \right) \in \prod_{j \in P_0} \mathcal{C}_j, \forall i \in [M] \setminus P_0 \ X_i \cap F = D_i \right\}} \right| > L_{P_0}. \quad (33)$$

Take now a product-permutation \mathfrak{L}_0 in which all sets D_i ($i \notin P_0$) and all elements of the chains \mathcal{C}_j ($j \in P_0$) are initial with respect to \mathfrak{L}_0 . Since $\mathbb{P}(\mathcal{F}_{S(I)}(\mathfrak{L}_0)) = T(I)$ we can rewrite (33) as

$$\sum_{\substack{(i_1, \dots, i_M) \in \pi_M: \\ \forall j \notin P_0 \ i_j = |D_j|}} t_{i_1, \dots, i_M}(I) > L_{P_0}. \quad (34)$$

As $T(I) = \mathbb{P}(\mathcal{H}_{\mathfrak{L}_0}(\mathfrak{L}_0))$, (34) gives

$$\left| \overline{\left\{ F^{\# [F, \mathcal{H}_{\mathfrak{L}_0}(\mathfrak{L}_0)]} : \left(F \cap \biguplus_{j \in P_0} X_j \right) \in \prod_{j \in P_0} \mathcal{C}_j, \forall i \in [M] \setminus P_0 \ X_i \cap F = D_i \right\}} \right| > L_{P_0}. \quad (35)$$

However, from $\mathcal{H}_{\mathfrak{L}_0} \in \mathbb{A}$ we get that $\mathcal{H}_{\mathfrak{L}_0}$, and consequently $\mathcal{H}_{\mathfrak{L}_0}(\mathfrak{L}_0)$ must be k -dimensional M -part Sperner multi-families with parameters $L_P : P \in \binom{[M]}{k}$, contradicting (35). \square

Proof to Lemma 5.4: Let \mathbb{A} be family of k -dimensional M -part Sperner multi-families that satisfy a Γ -multiplicity constraints \mathbb{M}_Γ , and let $I \in \pi_M$ be a k -dimensional multi-transversal with the same parameters L_P satisfying the same Γ -multiplicity constraint \mathbb{M}_Γ . Let \mathfrak{L} be a fixed product-permutation, for each $(i_1, \dots, i_M) \in I$ let $H_{(i_1, \dots, i_M)}$ be the (unique) initial set with respect to \mathfrak{L} with profile vector (i_1, \dots, i_M) and let

$$\mathcal{H}_{\mathfrak{L}} = \overline{\{ H_{(i_1, \dots, i_M)}^{t_{i_1, \dots, i_M}(I)} \}}.$$

It follows that $\mathcal{H}_{\mathfrak{L}}(\mathfrak{L}) = \mathcal{H}_{\mathfrak{L}}$, $\mathbb{P}(\mathcal{H}_{\mathfrak{L}}) = T(I)$, and from the properties of I we have that $\mathcal{H}_{\mathfrak{L}} \in \mathbb{A}$. Therefore we get that $T(I) \in \mu(\mathbb{A}(\mathfrak{L}))$. By Theorem 5.2, the vector $S(I)$ is present in the set on the right hand side of (11), whose extreme points agree with those of $\mu(\mathbb{A})$, and by Theorem 5.3, $S(I) \in \mu(\mathbb{A})$. All that remains to be shown is that if $S(I) = \sum_{T(I_u) \in \mu(\mathbb{A}(\mathfrak{L}))} \lambda(I_u) S(I_u)$ with $\lambda(I_u) \geq 0$ and $\sum_{T(I_u) \in \mu(\mathbb{A}(\mathfrak{L}))} \lambda(I_u) = 1$, then I is among the I_u 's, and all others come with a zero coefficient. $S(I) = \sum_{T(I_u) \in \mu(\mathbb{A}(\mathfrak{L}))} \lambda(I_u) S(I_u)$

means that for all $(i_1, \dots, i_M) \in \pi_M$ we have

$$t_{i_1, \dots, i_M}(I) \prod_{j=1}^M \binom{m_j}{i_j} = \sum_{T(I_u) \in \mu(\mathbb{A}(\mathfrak{L}))} \lambda(I_u) t_{i_1, \dots, i_M}(I_u) \prod_{j=1}^M \binom{m_j}{i_j},$$

which implies that

$$T(I) = \sum_{T(I_u) \in \mu(\mathbb{A}(\mathfrak{L}))} \lambda(I_u) T(I_u).$$

Let the ordering $\text{supp}(I) = \{\vec{j}_1, \vec{j}_2, \dots, \vec{j}_s\}$ show that I has the LEM property. Then for all u , $T_{\vec{j}_1}(I) \geq T_{\vec{j}_1}(I_u)$, and as the coefficients sum to 1, for all u , $T_{\vec{j}_1}(I) = T_{\vec{j}_1}(I_u)$. This argument repeats to $\vec{j}_2, \dots, \vec{j}_s$. Hence for all u , $\text{supp}(I) \subseteq \text{supp}(I_u)$. If $\text{supp}(I)$ is a proper subset of $\text{supp}(I_u)$, then we must have $\lambda(I_u) = 0$. Therefore for all the I_u that have $\lambda(I_u) \neq 0$ we must have $\text{supp}(I_u) = \text{supp}(I)$, and consequently $I_u = I$. \square

10 Proofs for the results on transversals

We start with two lemmata.

Lemma 10.1. *Let us be given n_1, n_2 positive integers, and set $\ell = \gcd(n_1, n_2)$, $m_i = \frac{n_i}{\ell}$ and $N = \text{lcm}(n_1, n_2) = \frac{n_1 n_2}{\ell}$. For every $j \in [N]^\star$, there are exactly ℓ vectors $(a_1, a_2) \in \pi_2$, such that $\left\langle \frac{a_1}{n_1} + \frac{a_2}{n_2} \right\rangle = \frac{j}{N}$.*

Proof. Since m_1, m_2 are relatively prime, for any integer $j \in [N]^\star$ we have integers z_1, z_2 such that $z_1 m_2 + z_2 m_1 = j$, therefore $\frac{z_1}{n_1} + \frac{z_2}{n_2} = \frac{j}{N}$. Taking $a_i \in [n_i]^\star$ such that $a_i \equiv z_i \pmod{n_i}$ we obtain that the required vectors (a_1, a_2) exist for any j . It is also clear that for any $(a_1, a_2) \in \pi_2$ there is some $j \in [N]^\star$ such that $\left\langle \frac{a_1}{n_1} + \frac{a_2}{n_2} \right\rangle = \frac{j}{N}$.

So we define for any $j \in [N]^\star$

$$\mathcal{D}_j = \left\{ (a_1, a_2) \in \pi_2 : \left\langle \frac{a_1}{n_1} + \frac{a_2}{n_2} \right\rangle = \frac{j}{N} \right\}.$$

Fix $j \in [N]^\star$ and $(x_1, x_2) \in \mathcal{D}_j$. For any $(y_1, y_2) \in \pi_2$ we have that $(y_1, y_2) \in \mathcal{D}_j$ iff $\frac{y_1 - x_1}{n_1} + \frac{y_2 - x_2}{n_2}$ is an integer.

The \mathcal{D}_j are nonempty and partition π_2 . If for each $j, j' \in [N]^\star$, there is an injection from \mathcal{D}_j to $\mathcal{D}_{j'}$, then $|\mathcal{D}_j| = |\mathcal{D}_{j'}|$, and consequently $|\mathcal{D}_j| = \frac{n_1 n_2}{N} = \ell$, which proves our statement. So we will construct such an injection.

Let $j, j' \in [N]^*$. Fix an $(a_1, a_2) \in \mathcal{D}_j$ and a $(b_1, b_2) \in \mathcal{D}_{j'}$. We define the map $\phi : \mathcal{D}_j \rightarrow \pi_2$ by $\phi(c_1, c_2) = (d_1, d_2) \in \pi_2$ iff $d_i \equiv c_i + (b_i - a_i) \pmod{n_i}$. Clearly, the map is a well-defined injection, moreover, $\phi(a_1, a_2) = (b_1, b_2)$.

Assume that $(d_1, d_2) \in \phi(\mathcal{D}_j)$. Then $(d_1, d_2) = \phi(c_1, c_2)$ for some $(c_1, c_2) \in \mathcal{D}_j$, and $d_i - b_i \equiv c_i - a_i \pmod{n_i}$. Thus $\left(\frac{d_1 - b_1}{n_1} + \frac{d_2 - b_2}{n_2}\right) - \left(\frac{c_1 - a_1}{n_1} + \frac{c_2 - a_2}{n_2}\right)$ is an integer. Since $(c_1, c_2) \in \mathcal{D}_j$, this implies $\frac{d_1 - b_1}{n_1} + \frac{d_2 - b_2}{n_2}$ is also an integer, with other words $(d_1, d_2) \in \mathcal{D}_{j'}$. Therefore $\phi(\mathcal{D}_j) \subseteq \mathcal{D}_{j'}$. \square

Lemma 10.2. *Let n_1, n_2, \dots, n_k be given, $K = \prod_{j=1}^k n_j$, $N = \text{lcm}(n_1, \dots, n_k)$ and $\ell = \frac{K}{N}$. For each $j \in [N]^*$ we have that there are exactly ℓ vectors $(a_1, \dots, a_k) \in \pi_k$ such that*

$$\left\langle \sum_{i=1}^k \frac{a_i}{n_i} \right\rangle = \frac{j}{N}.$$

Proof. We prove the statement by induction on k . The statement is clearly true for $k = 1$ (when $N = n_1$ and $\ell = 1$); and it was proved in Lemma 10.1 for $k = 2$. So assume that $k > 2$ and we know the statement already for all $1 \leq k' \leq k - 1$.

It is clear that for any $(a_1, \dots, a_k) \in \pi_k$ we have precisely one $j \in [N]^*$ such that $\left\langle \sum_{i=1}^k \frac{a_i}{n_i} \right\rangle = \frac{j}{N}$. Let $K_1 = \prod_{j=1}^{k-1} n_j$, $N_1 = \text{lcm}(n_1, \dots, n_{k-1})$ and $\ell_1 = \frac{K_1}{N_1}$, and $\ell_2 = \gcd(N_1, n_k)$. Then $K = K_1 n_k$, $N = \text{lcm}(N_1, n_k)$ and $\ell = \frac{K_1 n_k}{\text{lcm}(N_1, n_k)} = \frac{K_1}{N_1} \cdot \frac{N_1 n_k}{\text{lcm}(N_1, n_k)} = \ell_1 \ell_2$.

Fix a $j \in [N]^*$. Note that for integers a_i , $\left\langle \sum_{i=1}^{k-1} \frac{a_i}{n_i} \right\rangle \in \left\{ \frac{j'}{N_1} : j' \in [N_1]^* \right\}$, and for real numbers c, d we have $\langle \langle c \rangle + \langle d \rangle \rangle = \langle c + d \rangle$. By Lemma 10.1, there are precisely ℓ_2 pairs $(b, a_k) \in [N_1]^* \times [n_k]^*$ such that $\left\langle \frac{b}{N_1} + \frac{a_k}{n_k} \right\rangle = \frac{j}{N}$. By the induction hypothesis for each $b \in [N_1]$ there are precisely ℓ_1 values $(a_1, \dots, a_{k-1}) \in \pi_{k-1}$ such that $\left\langle \sum_{j=1}^{k-1} \frac{a_j}{n_j} \right\rangle = \left\langle \frac{b}{N_1} \right\rangle$. Since $\ell_1 \ell_2 = \ell$, the statement follows. \square

Proof to Lemma 7.4: By Lemma 10.2 the statement is equivalent with

$$\left| \left\{ j \in [N]^* : \left\langle \alpha + \frac{j}{N} \right\rangle \in [\beta, \beta + \mu) \right\} \right| \in \{ \lceil \mu N \rceil, \lfloor \mu N \rfloor \}$$

which follows from Lemma 7.1. \square

Proof to Theorem 7.5: Assume that μ satisfies condition (14) and $0 \leq \beta \leq 1 - \mu$. Fix $P \in \binom{[M]}{k}$ and for each $j \notin P$ fix a $b_j \in [n_j]$. Then Condition (1)

follows from Lemma 7.4 using $\alpha = \sum_{j \notin P} \frac{b_j}{n_j}$; thus $\mathbb{C}(n_1, \dots, n_M; \beta, \mu)$ is a k -dimensional transversal with the given parameters L_P .

Assume now further that for $P_0 \in \binom{[M]}{k}$ we have that $\mu = \frac{L_{P_0}}{K_{P_0}}$ (as this is equivalent with $\mu = \min_P \frac{L_P}{K_P}$). Then we have that

$$L_{P_0} \geq d_{P_0} \lceil \mu N_{P_0} \rceil = d_{P_0} \left\lceil \frac{L_{P_0}}{d_{P_0}} \right\rceil \geq L_{P_0},$$

which implies that μN_{P_0} is an integer, i.e. by Lemma 7.4 our transversal is full. \square

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